Lecture 3: Approximation Algorithms for Packing Problems in Two Dimensions

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at 10th Winter School on Computational Geometry
Amirkabir University of Technology, Tehran

2 March 2018

Strip Packing

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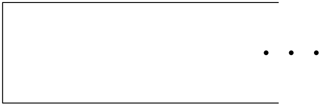

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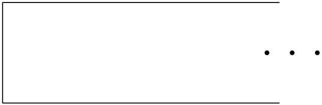

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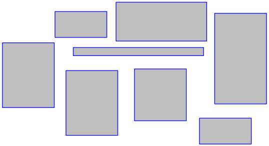

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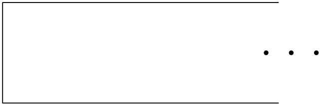

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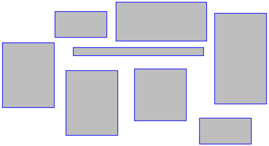

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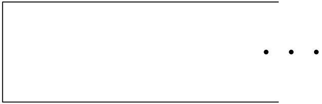

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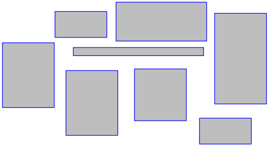

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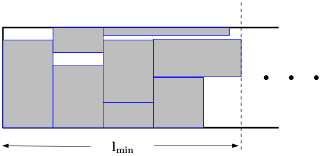

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both are approximation algorithms with factors 3 and 2.7 for FF and NF, respectively !

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2. Keep track of the rectangle areas obtained in step 1 and return the rectangle and packing achieving the minimum of these areas.

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- If there is an efficient α-approximation algorithm for strip packing then for any $\delta > 0$ there is an efficient $(1 + \delta)\alpha$-approximation algorithm for finding the minimum area a.p. rectangle for packing a.p. rectangles under translation.
- The best approximation algorithm for strip packing known achieves a factor of $5/3 + \varepsilon$ for any $\varepsilon > 0$ (Harren et al. 2014)
.
- For any $\varepsilon > 0$ there is an $5/3 + \varepsilon$-approximation algorithm for finding the smallest a.p. rectangle for packing a given set of rectangles under translation.

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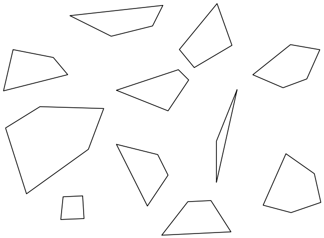

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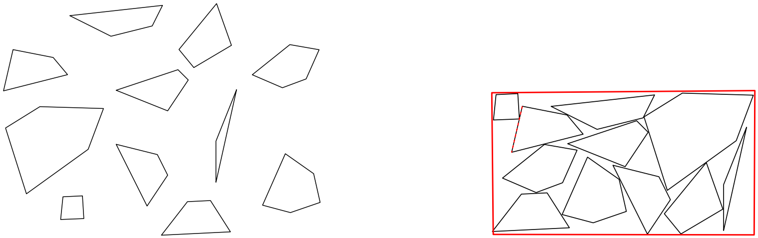

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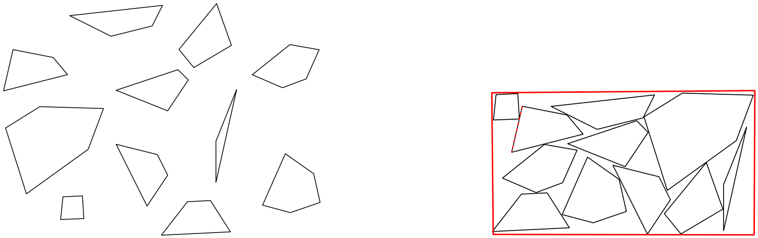

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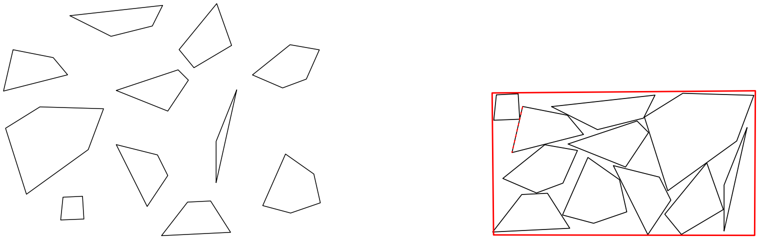

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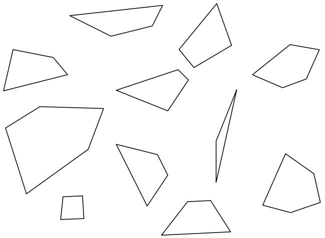

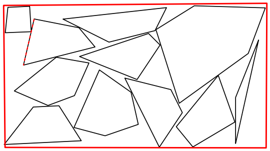

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- approximating the minimum area convex container for P under translations

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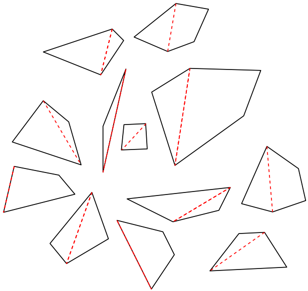

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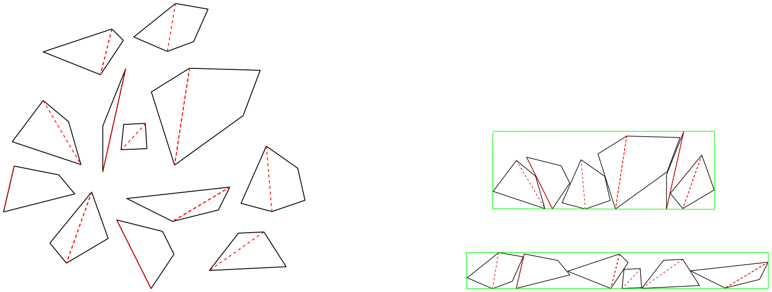

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3. cut the small boxes for the different height-classes into
   rectangles of equal width $2w_{max}$, stack all these rectangles,
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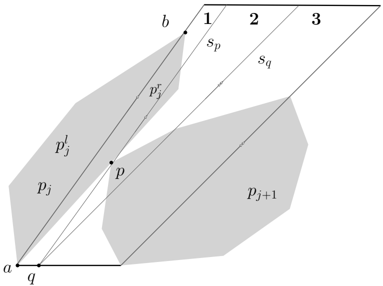

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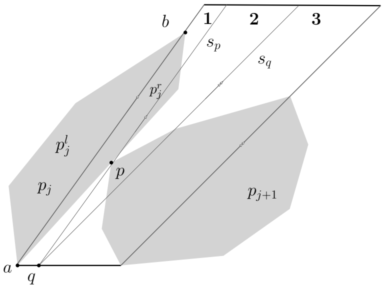

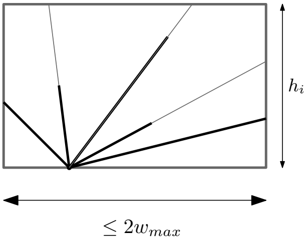

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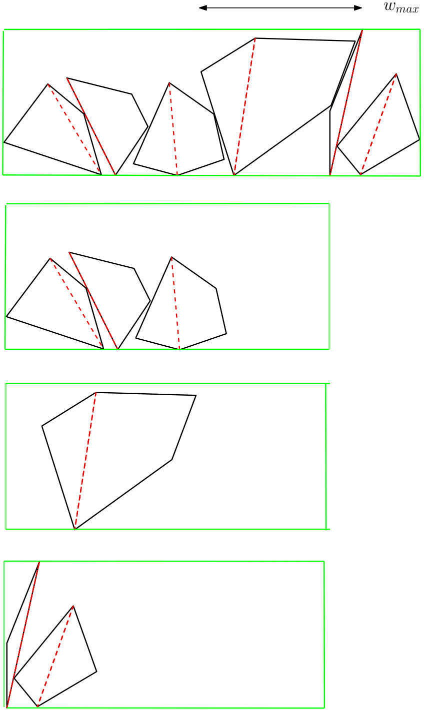
3. cut the small boxes for the different height-classes into rectangles of equal width $c w_{max}$, and stack all these rectangles, and return the resulting rectangle

gives an approximation factor of $f(\alpha, c) = \left(1 + \frac{1}{c}\right) \cdot \frac{2+c\alpha}{\alpha-\alpha^2}$

this is optimal for $\alpha = 0.407..$, $c = 2.214..$, namely 17.449...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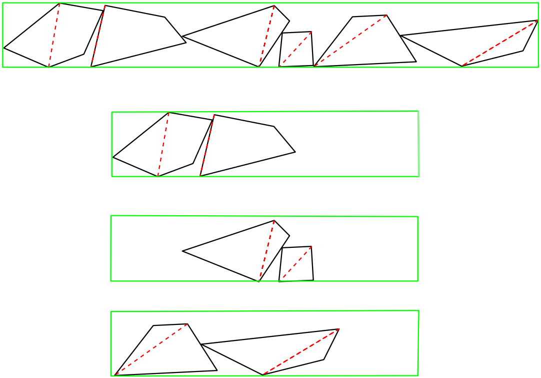

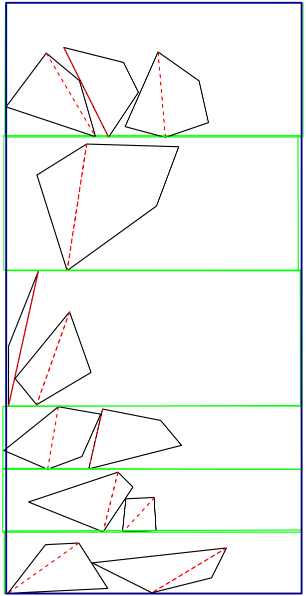

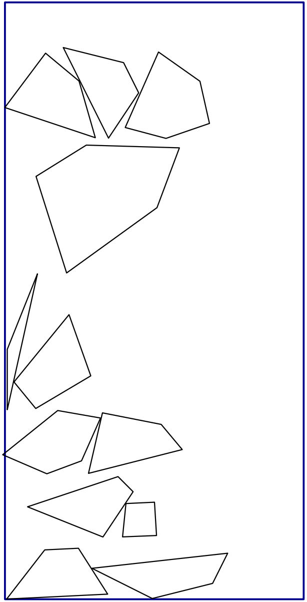

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2. Use the previous algorithm with respect to orientation ϕ.

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2. An approximation to the minimum area convex container can be found in $O(n \log n)$ time. Approximation factor: $16 + 8\sqrt{3} = 29.86..$