Lecture 5: Circle Packing 2

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Circle Packing Problems
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- **given:** radius \( r = \frac{p}{q} \in \mathbb{Q} \), \( p \) and \( q \) in binary
- **find:** the maximum number of unit circles which can be packed into a circle of radius \( r : C_r \)
Circle Packing Problems

• **given:** radius \( r = \frac{p}{q} \in \mathbb{Q} \), \( p \) and \( q \) in binary
  **find:** the maximum number of unit circles which can be packed into a circle of radius \( r : C_r \)

  or

• **given:** number \( n \in \mathbb{N} \) in binary
  **find:** the minimum radius \( r \) such that \( n \) unit circles can be packed into \( C_r \)
Circle Packing Problems

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find: the maximum number of unit circles which can be packed into a circle of radius \( r \): \( C_r \)

or

• given: number \( n \in \mathbb{N} \) in binary
find: the minimum radius \( r \) such that \( n \) unit circles can be packed into \( C_r \)

– extensively studied in mathematics for constant \( n \)
– already for \( n = 14 \) optimal solution unknown
Complexity of the Problems

Problem: short input lengths
Complexity of the Problems

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• in NP?
Complexity of the Problems

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- in NP? unknown
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- NP-hard?
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• in \( \text{NP} \) ? \quad \text{unknown}

• NP-hard ? \quad \text{unknown}

• in \( \exists R \) ?
Complexity of the Problems

Problem: short input lengths

- in $\text{NP}$? unknown
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- in $\exists R$? yes, for the first problem
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- in NP? unknown
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similar: pallet loading problem
Complexity of the Problems

Problem: short input lengths

- in \( \text{NP?} \) unknown
- NP-hard? unknown
- in \( \exists R ? \) yes, for the first problem, thus, in PSPACE

similar: pallet loading problem

we will show: PTAS for both variants
Infinite Packing of the Plane
packing as a **hexagonal grid** is optimal:
packing as a \textit{hexagonal grid} is optimal:

- density $\rho := \frac{\pi}{\sqrt{12}} \approx .907$
Infinite Packing of the Plane

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- density \( \rho := \frac{\pi}{\sqrt{12}} \approx 0.907 \)
- for lattices: Lagrange 1773
Infinite Packing of the Plane

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Infinite Packing of the Plane

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- rigorous proof by Fejes Tóth 1942

packing as a hexagonal grid is optimal:
Brute Force Algorithm

Decision Problem:
Can \( n \) unit circles be packed into a container circle with radius \( r = \frac{p}{q} \)?
Brute Force Algorithm

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formulated as Tarski-formula (even \( \exists R \)):

\[
\exists (x_1, \ldots, x_n, y_1, \ldots, y_n)
\prod_{1 \leq i < j \leq n} (x_i - x_j)^2 + (y_i - y_j)^2 - 4 \geq 0 \land \prod_{i=1}^{n} (x_i^2 + y_i^2 - (r - 1)^2 \leq 0)
\]
Brute Force Algorithm

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Brute Force Algorithm

Decision Problem:
Can $n$ unit circles be packed into a container circle with radius $r = \frac{p}{q}$?

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Formula can be solved in \( 2^{O(r^2 \log r)} L^3 \) time with algorithm by Basu et al. 1996
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input length
Brute Force Algorithm

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To solve optimization problem use binary search: \( 1 \leq n \leq r^2 \)

\( \rightarrow \) **running time** \( 2^{O(r^2 \log r)} L^3 \)
Input: Number $r > 3$, parameter $\varepsilon > 0$
Output: Integer $n(r) \geq 0$

if $r$ is small then
  Compute $n(r)$ brute force;
end

else

end
**PTAS**

<table>
<thead>
<tr>
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small means $r < c \cdot \frac{1}{\varepsilon}$

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$n(r) \geq \# \text{●}'s \geq \# \text{▲}'s/2$
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\[
n(r) \geq \# \bullet's \geq \# \triangle's/2
\]

\[
\geq \frac{\text{area}(C_{r-3})}{2 \cdot \text{area}(\triangle)}
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**Input:** rational number \( r = \frac{p}{q} > 3 \), parameter \( \varepsilon > 0 \)  
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\[ \geq \frac{\text{area}(C_{r-3})}{2 \cdot \text{area}(\triangle)} = \frac{\pi (r-3)^2}{\sqrt{12}} \]
Input: Number $r > 3$, parameter $\varepsilon > 0$
Output: Integer $n(r) \geq 0$
if $r < 13 \cdot \frac{1}{\varepsilon}$ then
    Compute $n(r)$ brute force;
else
    $n(r) = \left\lceil \frac{\pi (r-3)^2}{\sqrt{12}} \right\rceil$
end

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    $n(r) = \lceil \frac{\pi(r-3)^2}{\sqrt{12}} \rceil$
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We cannot compute these numbers exactly.
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We cannot compute these numbers exactly.

- compute $\pi$ and $\sqrt{12}$ only with the precision needed, depending on $r$
  → running time polynomial in input size
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- compute \( \pi \) and \( \sqrt{12} \) only with the precision needed, depending on \( r \)
  \( \rightarrow \) running time polynomial in input size
  \( \rightarrow \) total running time polynomial in input size
Upper Bound
packing where no circles can be added
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- Packing saturated: Triangles with some inner angle $\geq \frac{2\pi}{3}$ lie outside $C_{r-3}$. 
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Upper Bound

- Packing saturated: Triangles with some inner angle $\geq \frac{2\pi}{3}$ lie outside $C_{r-3}$.
- Density inside the union of all Delaunay triangles with at least one vertex in $C_{r-3}$:

$$\frac{\frac{1}{2} \cdot \text{area}(\bigcirc)}{\text{area}(\triangle)} \leq \frac{2\pi}{3}$$
Upper Bound

- Packing saturated: Triangles with some inner angle \( \geq \frac{2\pi}{3} \) lie outside \( C_{r-3} \).
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  \[
  \leq \frac{1}{2} \cdot \frac{\text{area}(\bigcirc)}{\text{area}(\triangle)} \leq \frac{2\pi}{3}
  \]
- Density of the rest of \( C_r \)
Upper Bound

- Packing saturated: Triangles with some inner angle $\geq \frac{2\pi}{3}$ lie outside $C_{r-3}$.
- Density inside the union of all Delaunay triangles with at least one vertex in $C_{r-3}$:
  $$\frac{\frac{1}{2} \cdot \text{area}(\bigcirc)}{\text{area}(\triangle)} \leq \frac{2\pi}{3}$$
- Density of the rest of $C_r$ \leq 1
- Gives together an upper bound of $n(r) \leq \frac{\pi}{\sqrt{12}} r^2 + 6r$
Upper Bound

- Packing saturated: Triangles with some inner angle $\geq \frac{2\pi}{3}$ lie outside $C_{r-3}$.
- Density inside the union of all Delaunay triangles with at least one vertex in $C_{r-3}$:
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  \frac{1}{2} \cdot \frac{\text{area} (\bigodot)}{\text{area} (\triangle)} \leq \frac{2\pi}{3}
  \]
- Density of the rest of $C_r$ \leq 1
- Gives together an upper bound of $n(r) \leq \frac{\pi}{\sqrt{12}}r^2 + 6r$
Putting it together
so: \[
\left\lfloor \frac{\pi (r-3)^2}{\sqrt{12}} \right\rfloor \leq n(r) \leq \frac{\pi}{\sqrt{12}} r^2 + 6r
\]
Putting it together

so: \[
\left\lfloor \frac{\pi(r-3)^2}{\sqrt{12}} \right\rfloor \leq n(r) \leq \frac{\pi}{\sqrt{12}} r^2 + 6r
\]

and \[
\frac{n_{\text{approx}}(r)}{n(r)} = \frac{\left\lfloor \frac{\pi(r-3)^2}{\sqrt{12}} \right\rfloor}{n(r)} \geq \frac{\left\lfloor \frac{\pi(r-3)^2}{\sqrt{12}} \right\rfloor}{\frac{\pi}{\sqrt{12}} r^2 + 6r} \geq 1 - \varepsilon,
\]
Putting it together

so: \[
\left\lceil \frac{\pi(r-3)^2}{\sqrt{12}} \right\rceil \leq n(r) \leq \frac{\pi}{\sqrt{12}} r^2 + 6r
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and \[
\frac{n_{approx}(r)}{n(r)} = \frac{\left\lceil \frac{\pi(r-3)^2}{\sqrt{12}} \right\rceil}{n(r)} \geq \frac{\left\lceil \frac{\pi(r-3)^2}{\sqrt{12}} \right\rceil}{\frac{\pi}{\sqrt{12}} r^2 + 6r} \geq 1 - \varepsilon,
\]

if \( r \geq \frac{12\sqrt{3}+6\pi}{\pi} \cdot \frac{1}{\varepsilon} \leq 13 \cdot \frac{1}{\varepsilon} \)
More PTAS

Given integer $n > 0$ find the smallest $r$ such that $n$ unit circles can be packed into a circle with radius $r$. 
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More PTAS

Given integer \( n > 0 \) find the smallest \( r \) such that \( n \) unit circles can be packed into a circle with radius \( r \).

- Problem cannot be formulated as an \( \exists R \)-formula, but as a Tarski formula.
More PTAS

Given integer $n > 0$ find the smallest $r$ such that $n$ unit circles can be packed into a circle with radius $r$.

- Problem cannot be formulated as an $\exists R$-formula, but as a Tarski formula.
- For $n < c \cdot \frac{1}{\varepsilon}$, solve this Tarski formula in time exponential in $n$. 
More PTAS

Given integer $n > 0$ find the smallest $r$ such that $n$ unit circles can be packed into a circle with radius $r$.

- Problem cannot be formulated as an $\exists R$-formula, but as a Tarski formula.
- For $n < c \cdot \frac{1}{\varepsilon}$, solve this Tarski formula in time exponential in $n$.
- For $n \geq c \cdot \frac{1}{\varepsilon}$, the bounds presented before can be transformed into bounds for this problem.
More PTAS

Given integer $n > 0$ find the smallest $r$ such that $n$ unit circles can be packed into a circle with radius $r$.

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- This gives together a PTAS for the reverse problem.
More PTAS

Given integer $n > 0$ find the smallest $r$ such that $n$ unit circles can be packed into a circle with radius $r$.

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- For $n \geq c \cdot \frac{1}{\varepsilon}$, the bounds presented before can be transformed into bounds for this problem.

- This gives together a PTAS for the reverse problem.

$\rightarrow$ PTAS for:
PTAS for the Smallest Container Circle

**Input:** Integer $n \geq 1$, parameter $\varepsilon > 0$

**Output:** Nonnegative rational number $r_{\text{approx}}$

\[
\text{if } n < \left( \frac{8}{\varepsilon} + 4 \right)^2 \text{ then compute } r_{\text{approx}} \text{ with the exact algorithm}
\]

\[
\text{else } k = \lfloor \log n \rfloor + 2
\]

\[
\text{compute some } a \text{ with } 12^{\frac{1}{4}} \sqrt{n} \leq a \leq 12^{\frac{1}{4}} \sqrt{n} + 2^{-k}
\]

\[
\text{compute some } b \text{ with } \sqrt{\pi} \geq b \geq \sqrt{\pi} - 2^{-k}
\]

\[
r_{\text{approx}} = \frac{a}{b} + 3
\]

\[
\text{return } r_{\text{approx}}
\]
3 Dimensions
3 Dimensions

Kepler’s conjecture (1611) and Hales’ theorem (2017)
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no arrangement of equally sized spheres filling space has a greater average density than that of the **cubic close packing** (face-centered cubic) and **hexagonal close packing** arrangements. The density of these arrangements is around 74.05%.

Wikipedia
Kepler’s conjecture (1611) and Hales’ theorem (2017)

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---

more specifically:

for every packing $V$, there exists a real number $c$ such that for every real number $r \geq 1$, the number of elements of $V$ contained in an open spherical container of radius $r$ centered at the origin is at most

$$\frac{\pi}{\sqrt{18}} \cdot r^3 + cr^2$$

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Hales 2015
Constant $c$ in Hales’ theorem needs to be made independent of the packing.
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That is possible: $c = 24373$. Scharf 2017
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