Delaunay triangulations of manifolds

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Delaunay triangulations of manifolds

1. Delaunay triangulations in Euclidean and Laguerre geometry
2. Good triangulations
3. Computational Topology
4. Shape reconstruction
5. Delaunay triangulation of manifolds

Bibliography

H. Edelsbrunner
Geometry and Topology for Mesh Generation, Cambridge 2001

T. Dey
Curve and surface reconstruction, Cambridge 2006

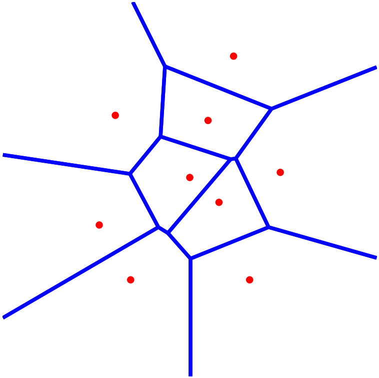
H. Edelsbrunner and J. Harer
Computational Topology, AMS 2010

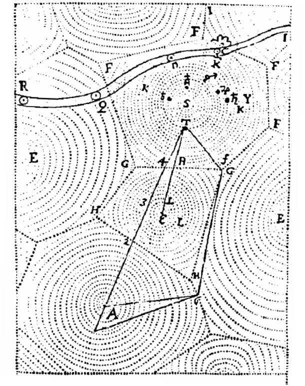
S-W. Cheng, T. Dey, J. Shewchuk
Delaunay Mesh Generation, CRC Press 2012.

J-D. Boissonnat, F. Chazal, M. Yvinec
Geometric and Topological Inference, Cambridge 2018

Voronoi diagrams

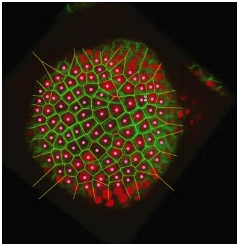
Fonction distance et croissance

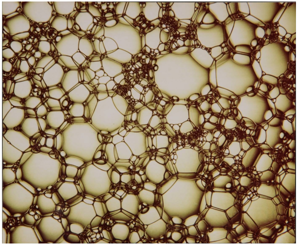




G. Voronoï
(1868-1908)

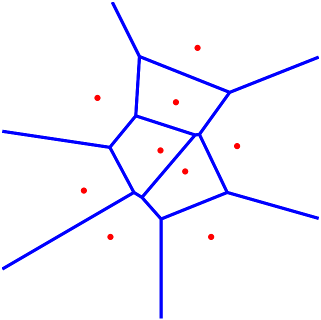
R. Descartes
(1596-1650)





Voronoi diagrams

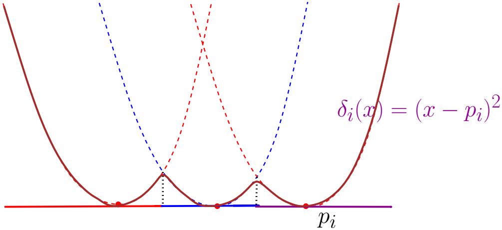
A set of points $\mathcal{P}$ in $\mathbb{R}^d$

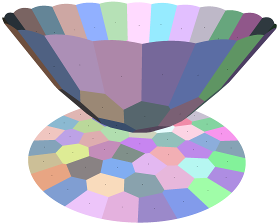


Voronoi cell $\quad V(p_i) = \{x : \|x - p_i\| \leq \|x - p_j\|, \ \forall j\}$

Voronoi diagram $(\mathcal{P})$ $\quad = \{$ set of cells $V(p_i)$, $p_i \in \mathcal{P}$ $\}$

Lower envelopes of functions





- $\mathrm{Vor}(p_i) = \{x : \delta_i(x) \leq \delta_j(x), \forall j\}$
- $\mathrm{Vor}(\mathcal{P})$ is the projection of the lower envelope of the δ_i
  = minimization diagram of the δ_i
- $\delta_i(x) \leq \delta_j(x) \quad \Leftrightarrow \quad h_{p_i} = p_i \cdot x - p_i^2 \geq h_{p_j} = p_j \cdot x - p_j^2$
- $\mathrm{Vor}(\mathcal{P})$ is the projection of the lower envelop of the h_{p_i}
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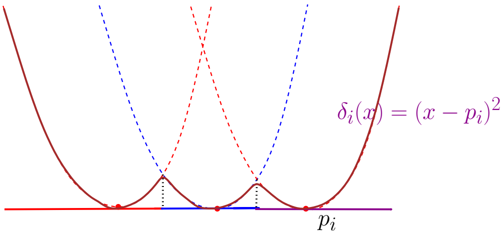
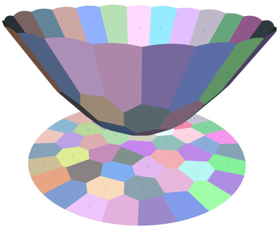
Lower envelopes of functions





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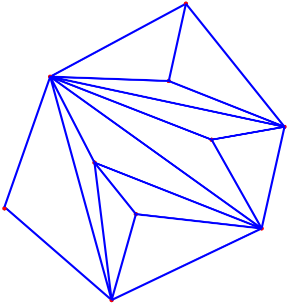
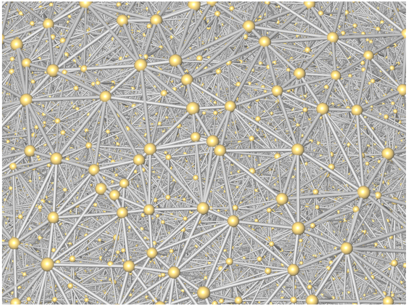
Triangulations of finite point sets of $\mathbb{R}^d$

Gluing simplices together



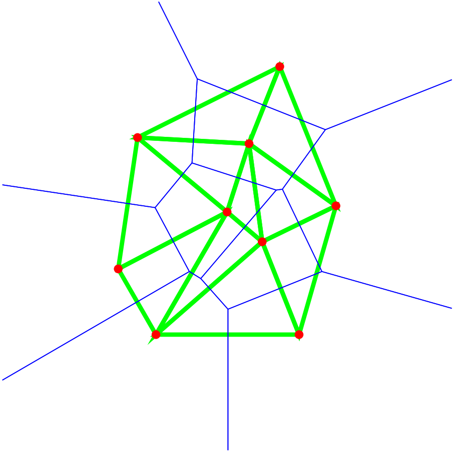


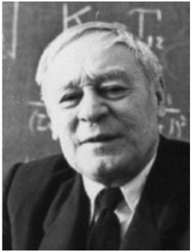
Triangulation of $\mathcal{P}$: a maximal set of d-simplices s.t.

- the intersection of two simplices is either empty or a common face of the two simplices
- the union of the simplices $= \mathrm{conv}(\mathcal{P})$

Delaunay Triangulations

Sur la sphère vide (On the empty sphere), Boris Delaunay (1934)





$\mathrm{Del}(\mathcal{P})$ is the nerve of $\mathrm{Vor}(\mathcal{P})$

Theorem

If no hypersphere contains $d+2$ points of $\mathcal{P}$, alors $\mathrm{Del}(\mathcal{P})$ is a triangulation of $\mathcal{P}$

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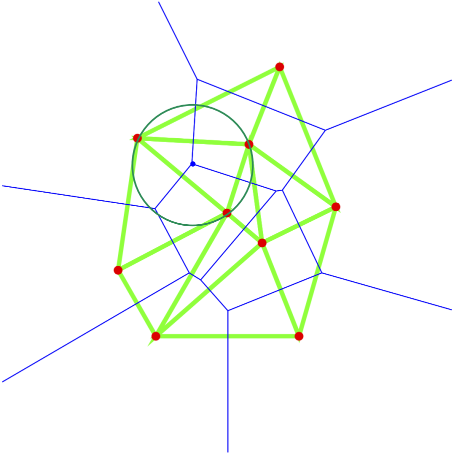

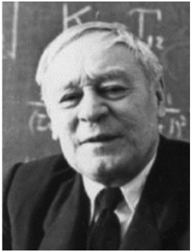

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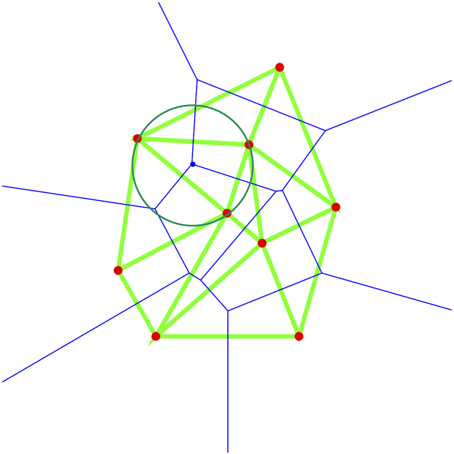

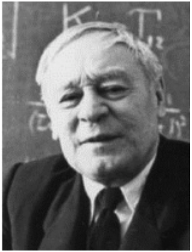

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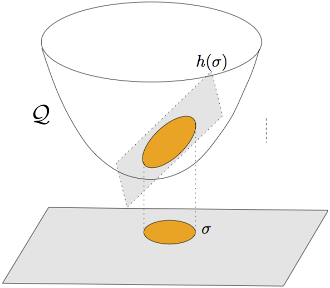
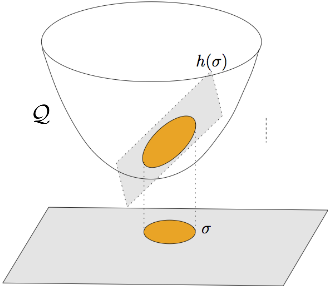


σ hypersphere of equation $\sigma(x) = 0$

$$\sigma(x) = x^2 - 2c \cdot x + s, \quad s = c^2 - r^2$$

$$\sigma(x) < 0 \Leftrightarrow \begin{cases} z < 2c \cdot x - s & (h_\sigma^-) \\ z = x^2 & (\mathcal{Q}) \end{cases}$$

$$\Leftrightarrow \hat{x} = (x, x^2) \in h_S^-$$



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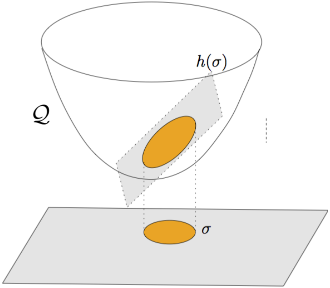

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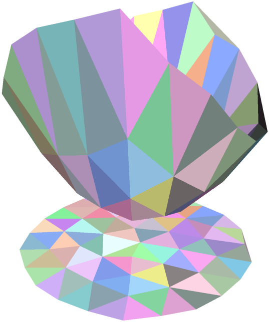
Proof of Delaunay's theorem

Delaunay's triangulation and convex hull

$\mathcal{P}$ in general position wrt spheres $\Leftrightarrow$ $\hat{\mathcal{P}}$ in general position



σ a simplex, B_σ its circumscribing ball

$\sigma \in \mathrm{Del}(\mathcal{P}) \Leftrightarrow \forall i,\ p_i \notin B_\sigma$

$\Leftrightarrow \forall i,\ \hat{p}_i \in h_\sigma^+ \ = \ \mathrm{aff}(\hat{\sigma})$

$\Leftrightarrow$ $\hat{\sigma}$ is a face of $\mathrm{conv}^-(\hat{\mathcal{P}})$

$$\mathrm{Del}(\mathcal{P}) = \mathrm{proj}(\mathrm{conv}^-(\hat{\mathcal{P}}))$$

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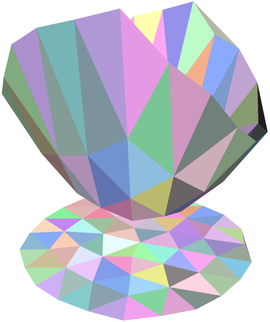

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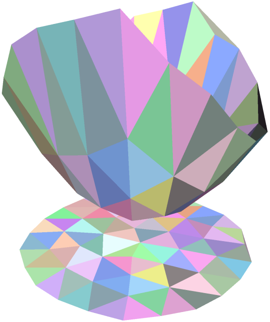

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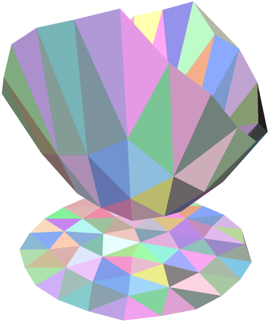

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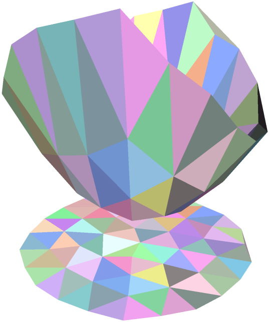

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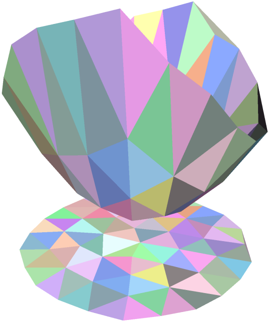

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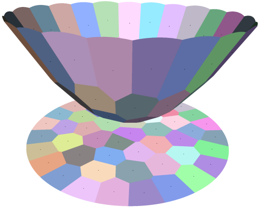
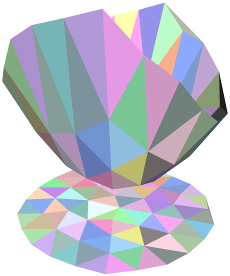
Correspondence between structures

$$h_{p_i} : x_{d+1} = 2p_i \cdot x - p_i^2 \qquad\qquad \hat{p}_i = (p_i, p_i^2) = h_{p_i}^*$$





$$\mathcal{V}(\mathcal{P}) = h_{p_1}^+ \cap \ldots \cap h_{p_n}^+ \quad \xrightarrow{\text{duality}} \quad \mathcal{D}(\mathcal{P}) = \text{conv}^-(\{\hat{p}_1, \ldots, \hat{p}_n\})$$

$$\uparrow \qquad\qquad\qquad\qquad \downarrow$$

Voronoi diagram of $\mathcal{P}$ $\xrightarrow{\text{nerve}}$ Delaunay triang. of $\mathcal{P}$

The diagram commutes if $\mathcal{P}$ is in general position wrt spheres

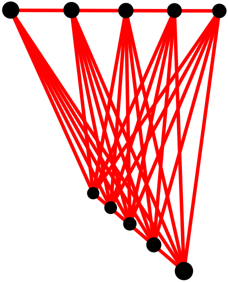
Happy consequences

Combinatorial complexity

The combinatorial complexity of the Voronoi diagram of n points of $\mathbb{R}^d$ is the same as the combinatorial complexity of the intersection of n half-spaces of $\mathbb{R}^{d+1}$

The combinatorial complexity of the Delaunay triangulation of n points of $\mathbb{R}^d$ is the same as the combinatorial complexity of the convex hull of n points of $\mathbb{R}^{d+1}$

The two complexities are the same by duality



$\Theta(n^{\lceil \frac{d}{2} \rceil})$

Quadratic in $\mathbb{R}^3$

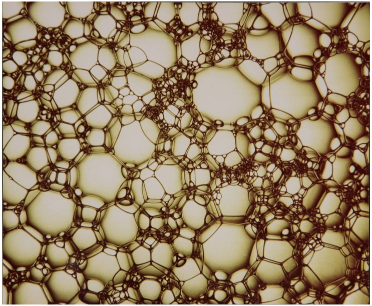
Construction of $\mathrm{Del}(\mathcal{P})$, $\mathcal{P} = \{p_1, ..., p_n\} \subset \mathbb{R}^d$

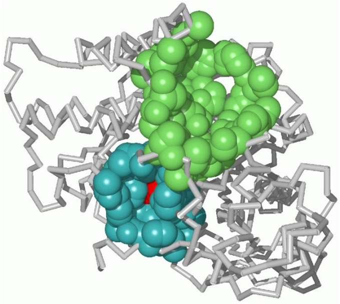
1. Lift the points of $\mathcal{P}$ onto the paraboloid $x_{d+1} = x^2$ of $\mathbb{R}^{d+1}$:
   $p_i \to \hat{p}_i = (p_i, p_i^2)$
2. Compute $\mathrm{conv}(\{\hat{p}_i\})$
3. Project the lower hull $\mathrm{conv}^-(\{\hat{p}_i\})$ onto $\mathbb{R}^d$

Complexity : $\Theta(n \log n + n^{\lfloor \frac{d+1}{2} \rfloor})$ [Clarkson & Shor 1989] [Chazelle 1993]

Foam and molecules

Diagrams of spheres and unions of balls





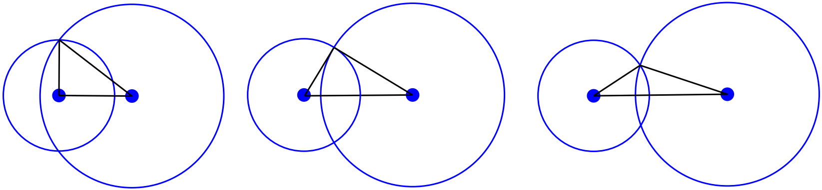
Orthogonal balls

Ball : $b(p,r) = \{x \in \mathbb{R}^d : \|p - x\| \leq r\}$

(Hyper)-sphere : $\partial b(p,r) = \{x \in \mathbb{R}^d : \|p - x\| = r\}$

« Distance » between balls : $D(b_1, b_2) = (p_1 - p_2)^2 - r_1^2 - r_2^2$

Orthogonal balls : $D(b_1, b_2) = 0$

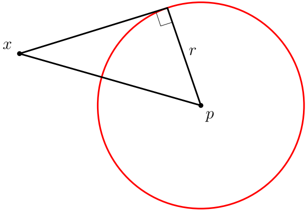


$D(b_1, b_2) < 0$ $D(b_1, b_2) = 0$ $D(b_1, b_2) > 0$

Power of a point wrt to a ball

Power of x wrt b : $D(x,b) = (x-p)^2 - r^2$

Z : D is not a distance

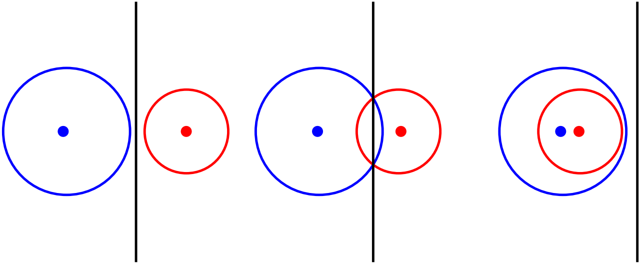


$$\begin{aligned} x \in \mathrm{int} b &\iff D(x,b) < 0 \\ x \in \partial b &\iff D(x,b) = 0 \\ x \notin b &\iff D(x,b) > 0 \end{aligned}$$

Radical hyperplane

- The set of points that have a same power wrt two balls $b_1(p_1, r_1)$ and $b_2(p_2, r_2)$ is a hyperplane

$$
\begin{aligned}
D(x, b_1) = D(x, b_2) \quad &\Longleftrightarrow \quad (x-p_1)^2 - r_1^2 \; = (x-p_2)^2 - r_2^2 \stackrel{\text{def}}{=} r_x^2 \\
&\Longleftrightarrow \quad -2p_1x + p_1^2 - r_1^2 = -2p_2x + p_2^2 - r_2^2 \\
&\Longleftrightarrow \quad 2(p_2 - p_1)x + (p_1^2 - r_1^2) - (p_2^2 - r_2^2) = 0
\end{aligned}
$$



- The radical hyperplane is the set of centres x of the balls $B(x, r_x)$ that are orthogonal to b_1 and b_2

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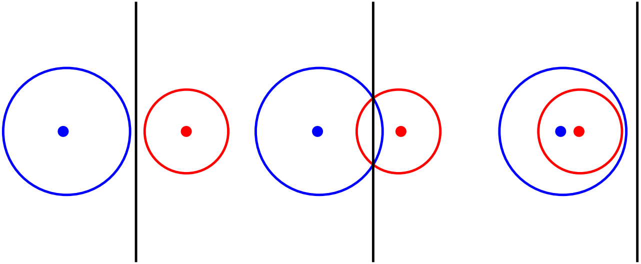

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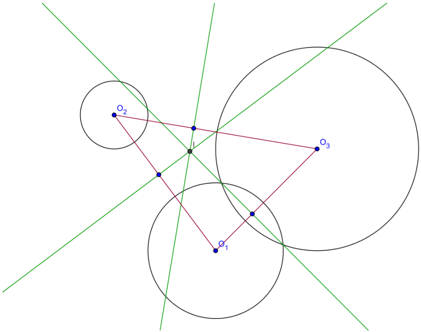
Radical centre



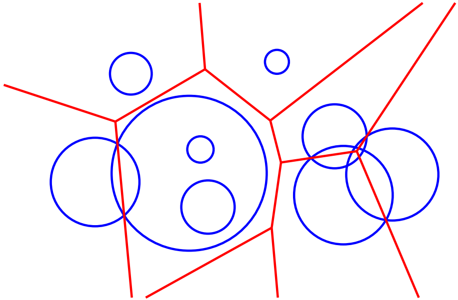
There exists a unique point with a same power wrt $d+1$ balls $b_0, ..., b_d$ of $\mathbb{R}^d$

this point is the centre of the unique ball that is orthogonal to $b_0, ..., b_d$

Set of balls $\mathcal{B}$ in general position : no ball is orthogonal to $d+2$ balls of $\mathcal{B}$

Laguerre (power) diagrams

$\mathcal{B} = \{b_1, ..., b_n\}$



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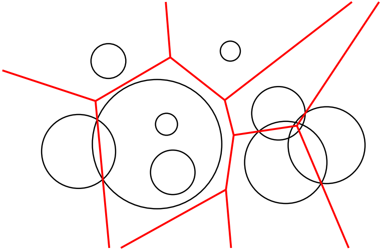
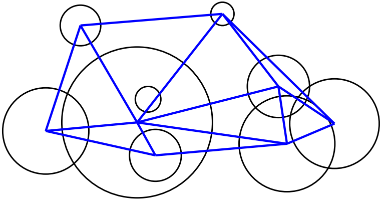
Voronoi diagram of $\mathcal{B}$: $=$ { set of cells $V(b_i), \ \ b_i \in \mathcal{B}\}$

Delaunay triangulations of balls



$\mathrm{Vor}(\mathcal{B})$

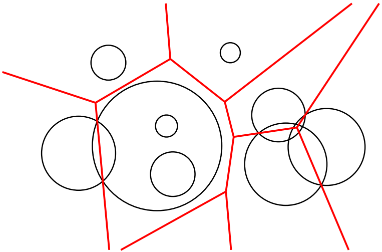


$\mathrm{Del}(\mathcal{B})$ is the nerve of $\mathrm{Vor}(\mathcal{B})$

Theorem

If the balls are in general position, then $\mathrm{Del}(\mathcal{B})$ is a triangulation of a subset $\mathcal{P}' \subseteq \mathcal{P}$ of the points

Delaunay triangulations of balls



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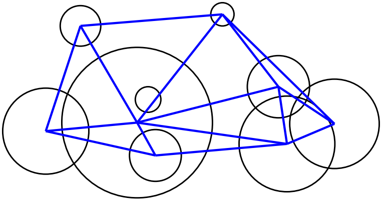

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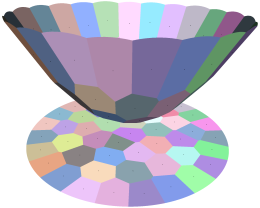
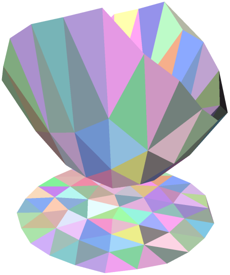
Correspondence between structures

$h_{b_i} : x_{d+1} = 2p_i \cdot x - p_i^2 + r_i^2$

$\hat{b}_i = (p_i, p_i^2 - r_i^2) = h^*_{b_i}$





$$\begin{array}{ccc}
\mathcal{V}(\mathcal{B}) = h^+_{b_1} \cap \ldots \cap h^+_{b_n} & \xrightarrow{\text{duality}} & \mathcal{D}(\mathcal{B}) = \mathsf{conv}^-(\{\hat{b}_1, \ldots, \hat{b}_n\}) \\
\uparrow & & \downarrow \\
\text{Voronoi diagram of } \mathcal{B} & \xrightarrow{\text{nerve}} & \text{Delaunay triang. of } \mathcal{B}
\end{array}$$

The diagram commutes if $\mathcal{B}$ is in general position

Affine diagrams

Sites + distance functions s.t. the bisectors are hyperplanes

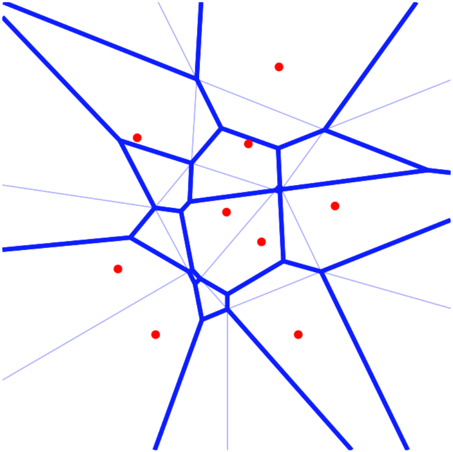
Theorem [Aurenhammer]

Any affine diagram of $\mathbb{R}^d$ is the Laguerre diagram of a set of balls of $\mathbb{R}^d$

Example : The intersection of a Voronoi diagram with an affine space is a Laguerre diagram

Voronoi diagram of order k

An example of an affine diagram



Each cell is the set of points that have the same k nearest sites

Voronoi diagrams of order k are Laguerre diagrams

$S_1, S_2, \ldots$ the subsets of k points of $\mathcal{P}$

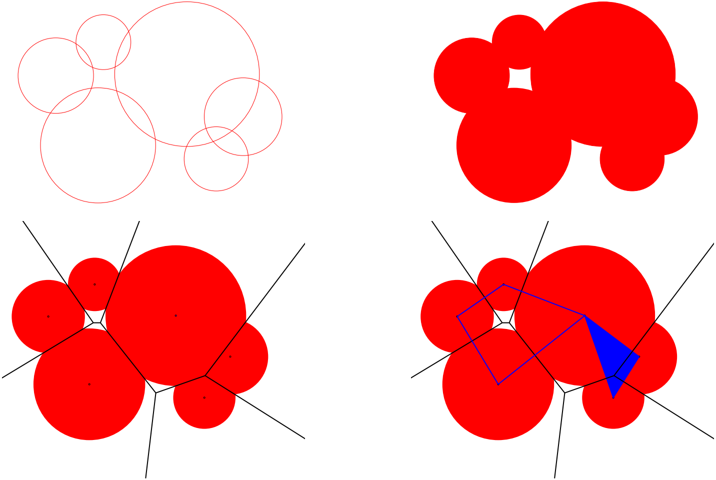
$$\begin{aligned}\delta(x, S_i) &= \frac{1}{k} \sum_{p \in S_i} (x-p)^2 \\ &= x^2 - \frac{2}{k} \sum_{p \in S_i} p \cdot x + \frac{1}{k} \sum_{p \in S_i} p^2 \\ &= D(b_i, x)\end{aligned}$$

where b_i is the ball centered at $c_i = \frac{1}{k} \sum_{p \in S_i} p$

of radius $r_i^2 = c_i^2 - \frac{1}{k} \sum_{p \in S_i} p^2$

$$x \in \mathrm{Vor}_k(S_i) \quad \Leftrightarrow \quad \delta(x, S_i) \leq \delta(x, S_j) \ \forall j$$

Delaunay triangulation restricted to a molecule



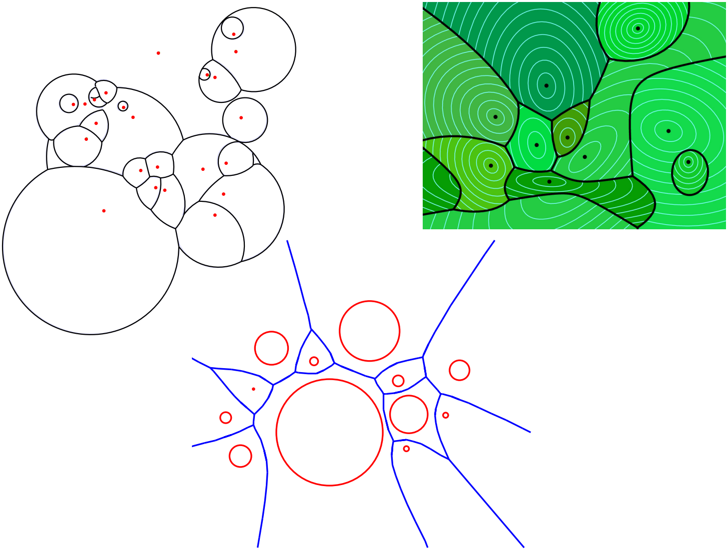
$C(b) = b \cap V(b)$

$U = \bigcup_{b \in \mathcal{B}} C(b)$

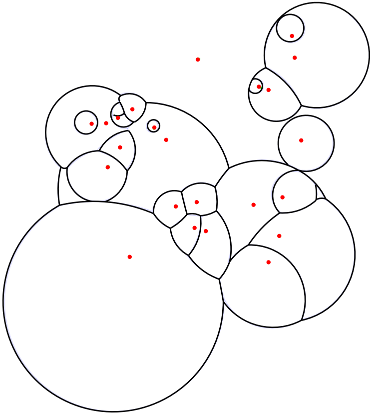
$\mathrm{Vor}_{|U}(\mathcal{B}) = \{f \in \mathrm{Vor}(\mathcal{B}), \quad f \cap U \neq \emptyset\}$

$\mathrm{Del}_{|U}(\mathcal{B})$

Distance functions and growth models



Möbius diagrams



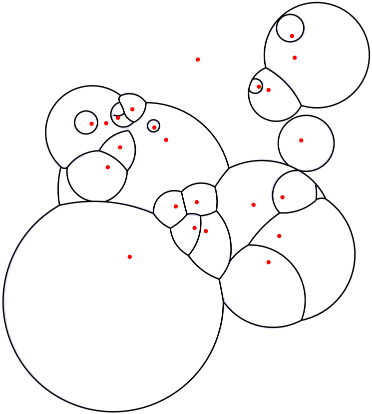
$$W_i = (p_i, \lambda_i, \mu_i)$$

$$\delta_M(x, W_i) = \lambda_i \|x - p_i\|^2 - \mu_i$$

$$\mathrm{Mob}(W_i) = \{x, \delta(x, \sigma_i) \leq \delta(x, \sigma_j)\}$$

Bisectors are hyperspheres (hyperplanes or $\emptyset$)

Möbius diagrams



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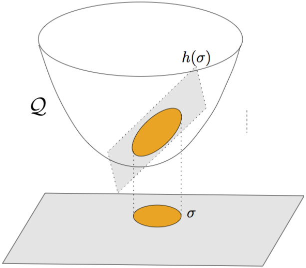
Bisectors are hyperspheres (hyperplanes or $\emptyset$)

Linearization

A Möbius diagram of $\mathbb{R}^d$ is the restriction of an affine diagram of $\mathbb{R}^{d+1}$



Lift the spherical bisectors onto $\mathcal{Q}$ and take the polar hyperplanes

The hyperplanes define an affine diagram $\mathrm{Vor}(\mathcal{B})$ in $\mathbb{R}^{d+1}$

The faces of the Möbius diagram are the projections of the faces of $\mathrm{Vor}(\mathcal{B}) \cap \mathcal{Q}$

Corollaries

1. *Any spherical diagram (i.e. whose bisectors are hyperspheres) is a Möbius diagram*
2. *The set of Möbius diagrams is stable under Möbius transformation*
3. *The intersection of a spherical diagram with an affine subspace is a spherical diagram*
4. $\boxed{Z}$ *: the nerve of a Möbius diagram has a realization in $\mathbb{R}^{d+1}$ but is not (in general) a triangulation of $\mathbb{R}^d$*

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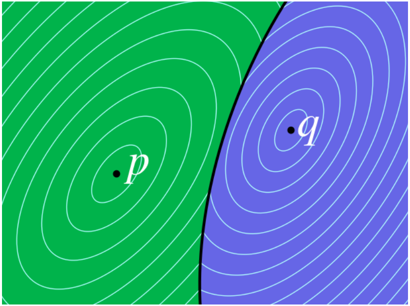
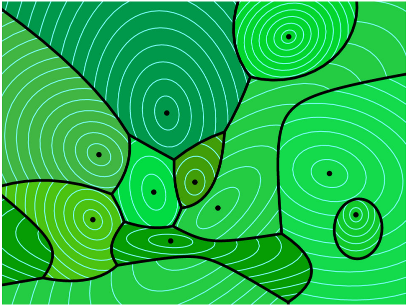
Anisotropic Voronoi diagrams

Metric at p : M_p : $d \times d$ matrix that is symetric, positive definite

$$d_p(x, y) = \sqrt{(x-y)^t \, M_p \, (x-y)}$$





[Labelle & Shewchuk 2003]

$$V(p) = \{x : d_p(x,p) \le d_q(x,q) \text{ for all } p, q \in P\}$$

Linearization

To any point $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$, associate the two points

$$\begin{aligned} \tilde{x} &= (x_i x_j, 1 \le i \le j \le d) \in \mathbb{R}^{\frac{d(d+1)}{2}} \\ \hat{x} &= (x, \tilde{x}) \in \mathbb{R}^D, \;\; D = \frac{d(d+3)}{2} \end{aligned}$$

- Observe that $\mathcal{Q} = \{\hat{x},\ x \in \mathbb{R}^d\}$ is a « surface » of dimension d in $\mathbb{R}^D$

- By elementary calculations : $d_p(x,p)^2 = -2\,\hat{p}^t\,\hat{x} + p^t\,M_p\,p$

  which implies

  $$d_p(x,p) < d_q(x,q) \quad \Leftrightarrow \quad (\hat{x}-\hat{p})^2 - (\hat{p}^2 - p^t\,M_p\,p) < (\hat{x}-\hat{q})^2 - (\hat{q}^2 - q^t\,M_q\,q)$$

Theorem

The anisotropic Voronoi diagram of $\mathcal{P}$ is the projection of the restriction of the Laguerre diagram of a set of n balls restricted to $\mathcal{Q}$

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A point represents a probability density function (pdf), for example the isotropic gaussian defined in $\mathbb{R}^d$

$$f(x, \mu, \sigma) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(\frac{-\|x-\mu\|}{2\sigma^2}\right)$$

can be represented by the point (μ, σ) in the space

$$H = \{(\mu, \sigma) \in \mathbb{R}^{d+1}, \sigma > 0\}$$

- What distance in those spaces ?
- Can we define and construct Voronoi diagrams in statistical spaces ?

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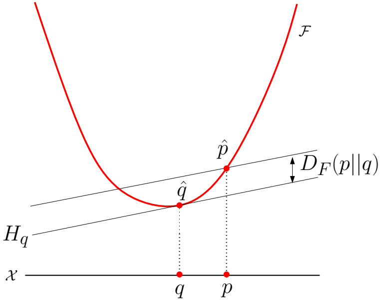
Bregman divergences

F a strictly convex and differentiable function defined on a convex set $\mathcal{X}$

$$D_F(\mathbf{p}, \mathbf{q}) = F(\mathbf{p}) - F(\mathbf{q}) - \langle \mathbf{p} - \mathbf{q}, \boldsymbol{\nabla}_F(\mathbf{q}) \rangle$$



D_F is not a distance but $D_F(\mathbf{p}, \mathbf{q}) \geq 0$ and $D_F(\mathbf{p}, \mathbf{q}) = 0$ iff $\mathbf{p} = \mathbf{q}$

Examples of Bregman divergences

- $F(x) = x^2$: Squared euclidean distance

  $$D_F(\mathbf{p},\mathbf{q}) = F(\mathbf{p}) - F(\mathbf{q}) - \langle \mathbf{p}-\mathbf{q}, \nabla_F(\mathbf{q})\rangle = \mathbf{p}^2 - \mathbf{q}^2 - \langle \mathbf{p}-\mathbf{q}, 2\mathbf{q}\rangle = \|\mathbf{p}-\mathbf{q}\|^2$$

- $F(p) = \sum p(x)\log_2 p(x)$ (Shannon entropy)
  $D_F(p,q) = \sum_x p(x)\log_2 \frac{p(x)}{q(x)}$ (K-L divergence)

- $F(p) = -\sum_x \log p(x)$ (Burg entropy)
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Bregman diagrams

[Boissonnat, Nielsen, Nock 2010]

$$D_F(\mathbf{p},\mathbf{q}) = F(\mathbf{p}) - F(\mathbf{q}) - \langle \mathbf{p}-\mathbf{q}, \boldsymbol{\nabla}_F(\mathbf{q})\rangle$$

Two types of bisectors

$$\begin{aligned} H_{pq} : D_F(\mathbf{x},\mathbf{p}) &= D_F(\mathbf{x},\mathbf{q}) \quad \text{(hyperplane)} \\ H^*_{pq} : D_F(\mathbf{p},\mathbf{x}) &= D_F(\mathbf{q},\mathbf{x}) \quad \text{(hypersurface)} \end{aligned}$$

Bregman diagrams

- Two types of Bregman diagrams
- By the Legendre duality : $D_F(\mathbf{x},\mathbf{y}) = D_{F^*}(\mathbf{y}',\mathbf{x}')$ $\qquad (\mathbf{x}' = \nabla_F(x))$

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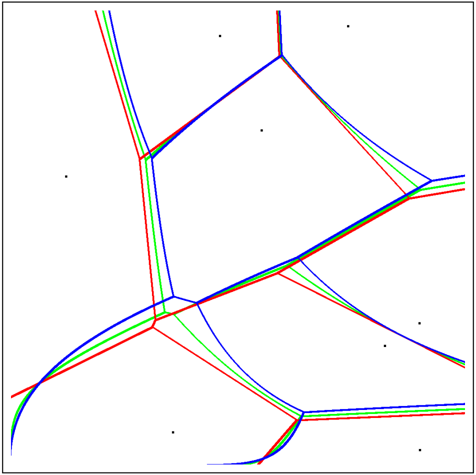

Bregman diagrams and Laguerre diagrams

The Bregman diagram of the 1st type of a set of n sites of $\mathcal{P}$ is identical to the Laguerre diagram of n euclidian balls centered at the points $\mathbf{p}'_i$

$$
\begin{aligned}
& D_F(\mathbf{x}, \mathbf{p}_i) \le D_F(\mathbf{x}, \mathbf{p}_j) \\
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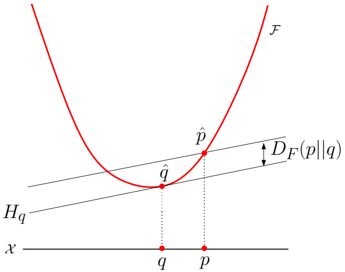
Bregman spheres

Definition : $\sigma(\mathbf{c}, r) = \{\mathbf{x} \in \mathcal{X} \mid D_F(\mathbf{x}, \mathbf{c}) = r\}$

Lemma The image $\hat{\sigma}$ of a Bregman sphere σ by the lifting map onto $\mathcal{F}$ is contained in a hyperplane H_σ

Conversely, the intersection of any hyperplane H with $\mathcal{F}$ projects vertically onto a Bregman sphere



Lemma A d-simplex has a unique circumscribing Bregman hypersphere

Bregman triangulations

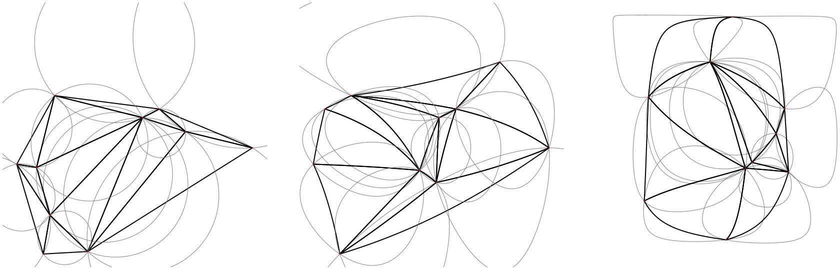
Definition : the Bregman triangulation of $\mathcal{P}$, $BT(\mathcal{P})$ is the nerve of

the Bregman diagram of $\mathcal{P}$ of the 1st type

and therefore also of a Laguerre diagram of $\mathcal{P}'$

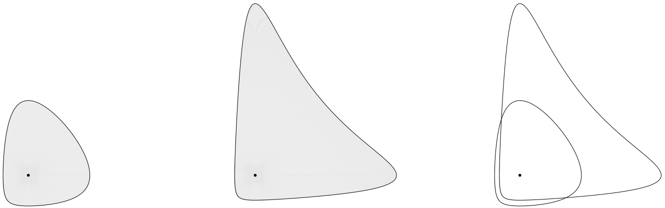
Characteristic property : The Bregman sphere circumscribing any simplex of $BT(\mathcal{P})$ is empty

Examples



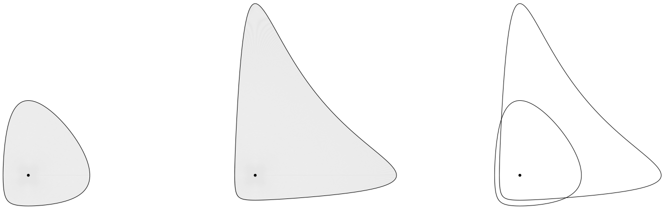
(a) Ordinary Delaunay (b) Exponential loss (c) Hellinger-like divergence

Unions of Bregman balls of types 1 and 2



- The combinatorial and algorithmic complexity of a union of Bregman balls is the same as for euclidean balls
- The same is true for unions of balls of the 2nd type (via Legendre transform which is a homeomorphism (if F is a Legendre function)

Unions of Bregman balls of types 1 and 2



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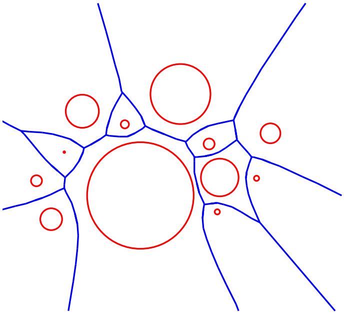
Next lectures

- Practical combinatorial and algorithmic complexity
- More general metrics
- Triangulation of surfaces and other curved spaces

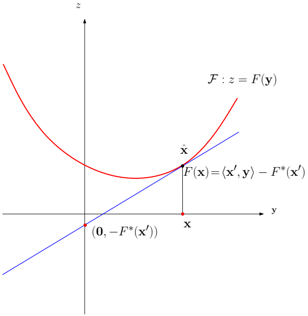
Open question

- Combinatorial complexity of the additively weighted Voronoi diagram



Legendre duality

Convex conjugate : $F^*(x') = x \cdot x' - F(x)$



Gradient space : $\Omega' = \{\nabla F(x), x \in \Omega\}$

F is a function of Legendre type if Ω' is convex

- $(F^*)^* = F$
- F^* is strictly convex and differentiable
- Writing $\mathbf{y}' = \boldsymbol{\nabla}_F(\mathbf{y})$: $$F^*(\mathbf{y}') = -F(\mathbf{y}) + \langle \mathbf{y}, \mathbf{y}' \rangle$$ $$\boldsymbol{\nabla}_F^* = \boldsymbol{\nabla}_F^{-1}$$

$$\begin{aligned} D_F(\mathbf{x}, \mathbf{y}) &= F(\mathbf{x}) - F(\mathbf{y}) - \langle \mathbf{x} - \mathbf{y}, \mathbf{y}' \rangle \\ &= -F^*(\mathbf{x}') + \langle \mathbf{x}, \mathbf{x}' \rangle + F^*(\mathbf{y}') - \langle \mathbf{x}, \mathbf{y}' \rangle \\ &= D_{F^*}(\mathbf{y}', \mathbf{x}') \end{aligned}$$