Delaunay triangulation of manifolds

3. Triangulation of topological spaces

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Topological spaces

A topology on a set X is a family $\mathcal{O}$ of subsets of X that satisfies the three following conditions :

1. the empty set $\emptyset$ and X are elements of $\mathcal{O}$,
2. any union of elements of $\mathcal{O}$ is an element of $\mathcal{O}$,
3. any finite intersection of elements of $\mathcal{O}$ is an element of $\mathcal{O}$.

The set X together with the family $\mathcal{O}$, whose elements are called open sets, is a topological space.

Homeomorphism

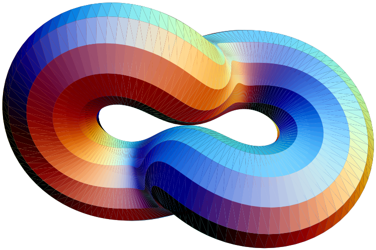
$f : X \to Y$ is a bijective mapping that is continuous and has a continuous inverse

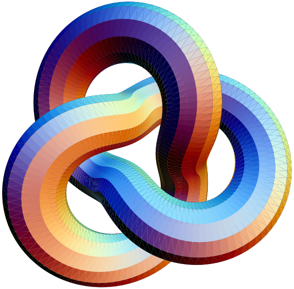
$$X \approx Y$$

Embedding

If $f : X \to Y$ is a homeomorphism onto its image, f is called an embedding of X into Y

Are these objects homeomorphic ?





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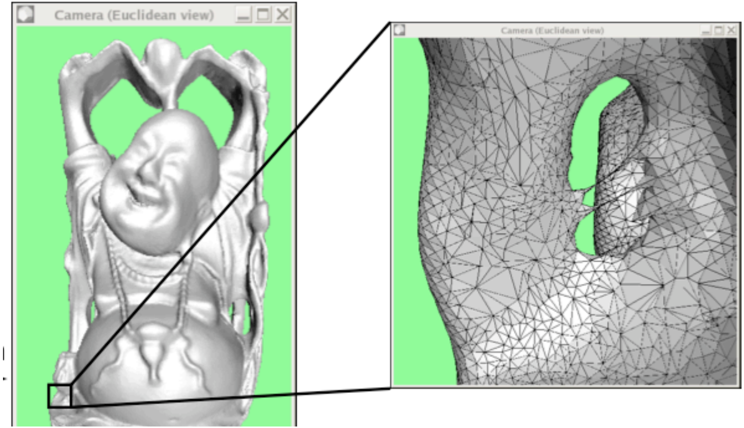
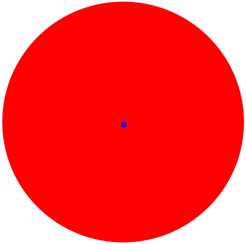
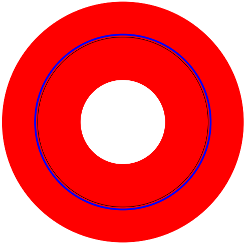


Two continuous mappings $f_0, f_1 : X \to Y$ are homotopic
if there exists a continuous mapping $h : [0,1] \times X \to Y$ s.t.

$$\forall x \in X, \quad h(0,x) = f_0(x) \quad \text{et} \quad h(1,x) = f_1(x)$$





Deformation retract : $f : X \to Y \subseteq X$ is a deformation retract if f is homotopic to the identity

Continuous mappings between topological spaces
Homotopy

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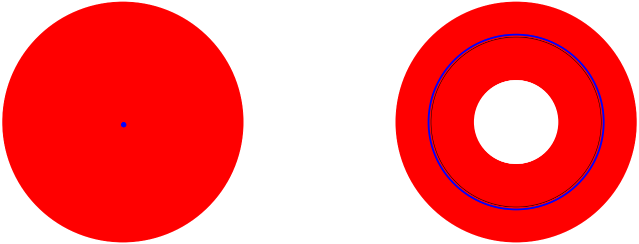

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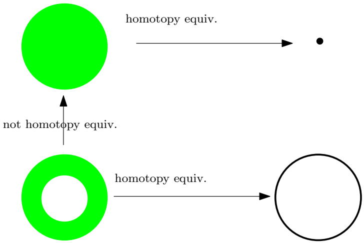
Continuous mappings between topological spaces

Homotopy equivalence



X and Y have the same homotopy type ($X \simeq Y$) if there exists two continuous mappings $f : X \to Y$ and $g : Y \to X$ s.t.

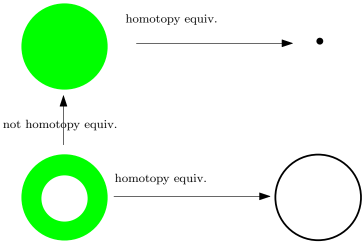
$f \circ g$ is homotopic to the identity mapping in Y

$g \circ f$ is homotopic to the identity mapping in X

X is contractible if it has the same homotopy type as a point

Continuous mappings between topological spaces

Homotopy equivalence



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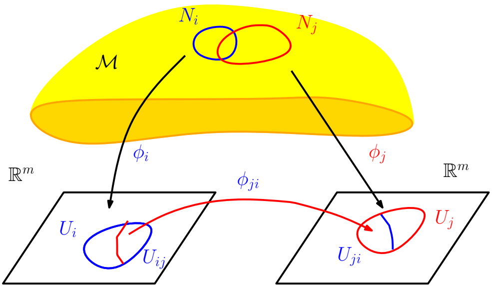
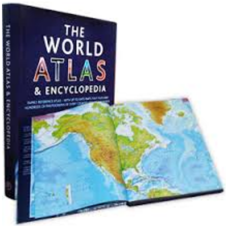
Curves, surfaces and manifolds

Charts, atlases and transition maps





Manifold : X is a manifold without boundary of dimension k if any $x \in X$ has a neighborhood that is homeomorphic to an open ball of dimension k of $\mathbb{R}^k$

Chart : ϕ_i homeomorphism

Transition map : ϕ_{ij} mapping between charts

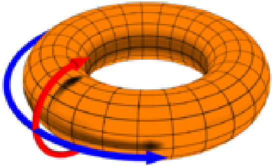
Example : Configuration spaces of mechanisms

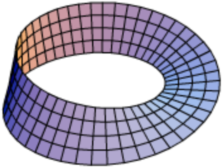
Intrinsic dimension and embedding

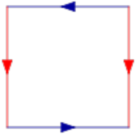
Whitney's embedding theorem

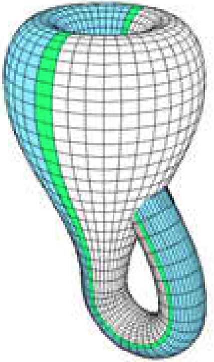
Any manifold of dimension k can be embedded in $\mathbb{R}^{2k+1}$

Some surfaces like the Klein bottle
cannot be embedded in $\mathbb{R}^3$



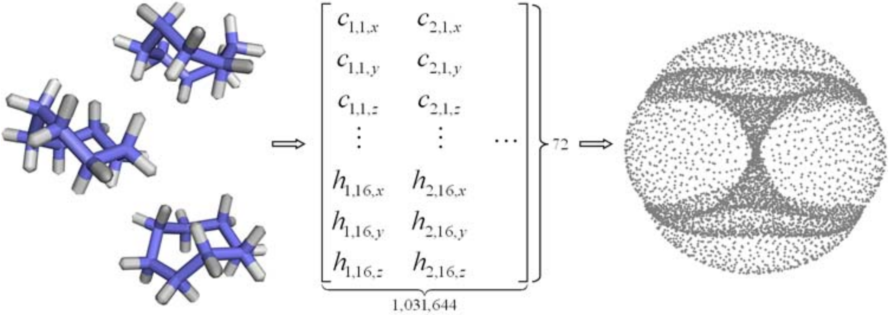






The configuration space of cyclo-octane C_8H_{16}

Stratified manifolds



Martin et al. [2010]

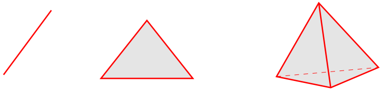
Geometric simplices

A k-simplex σ is the convex hull of $k+1$ points of $\mathbb{R}^d$ that are affinely independent

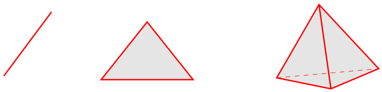
$$\sigma = \text{conv}(p_0, ..., p_k) = \{x \in \mathbb{R}^d,\ x = \sum_{i=0}^{k} \lambda_i\, p_i, \quad \lambda_i \in [0,1],\ \ \sum_{i=0}^{k} \lambda_i = 1\}$$

$k = \dim(\text{aff}(\sigma))$ is called the dimension of σ

1-simplex = line segment
2-simplex = triangle
3-simplex = tetrahedron



Faces of a simplex



$V(\sigma) =$ set of vertices of a k-simplex σ

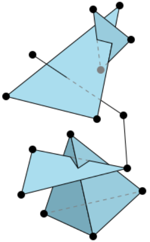
$\forall V' \subseteq V(\sigma)$, $\mathrm{conv}(V')$ is a face of σ

a k-simplex has $\begin{pmatrix} k+1 \\ i+1 \end{pmatrix}$ faces of dimension i

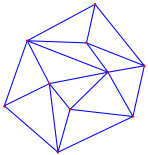
total nb of faces $= \sum_{i=0}^{d} \begin{pmatrix} k+1 \\ i+1 \end{pmatrix} = 2^{k+1} - 1$

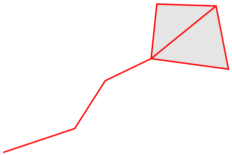
Geometric simplicial complexes

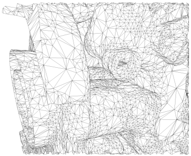
A finite collection of simplices K called the faces of K such that



- $\forall \sigma \in K$, σ is a simplex
- $\sigma \in K, \tau \subset \sigma \Rightarrow \tau \in K$
- $\forall \sigma, \tau \in K$, either $\sigma \cap \tau = \emptyset$ or $\sigma \cap \tau$ is a common face of both







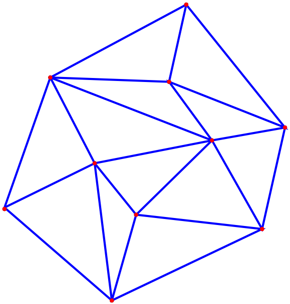
Geometric simplicial complexes

The dimension of a simplicial complex K is the max dimension of its simplices

A subset of K which is a complex is called a subcomplex of K

The underlying space $|K| \subset \mathbb{R}^d$ of K is the union of the simplices of K

Example 1 : Triangulation of a finite point set of $\mathbb{R}^d$



- A simplicial d-complex K is pure if every simplex in K is the face of a d-simplex.

- A triangulation of a finite point set $\mathcal{P} \in \mathbb{R}^d$ is a pure geometric simplicial complex K s.t. $\text{vert}(K) = \mathcal{P}$ and $|K| = \text{conv}(\mathcal{P})$.

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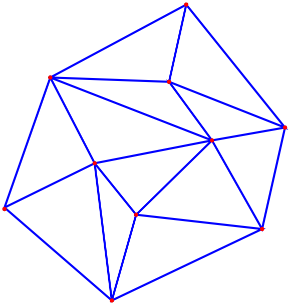

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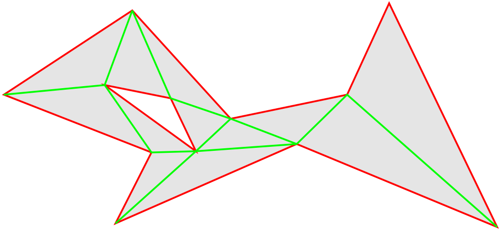
Example 2 : triangulation of a polygonal domain of $\mathbb{R}^2$



A triangulation of a polygonal domain $\Omega \subset \mathbb{R}^2$ is a pure geometric simplicial complex K s.t. $\quad \mathrm{vert}(K) = \mathrm{vert}(\Omega) \quad$ and $\quad |K| = \Omega$.

Basic facts

- Any bounded polygonal domain $\Omega \subset \mathbb{R}^2$ admits a triangulation
- Such a triangulation can be computed in time $O(n \log n)$ where $n = \sharp \mathrm{vert}(\Omega)$
- Some polyhedral domains of $\mathbb{R}^3$ do not admit a triangulation

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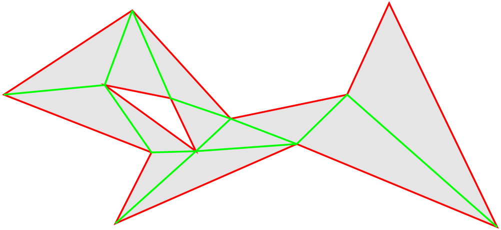

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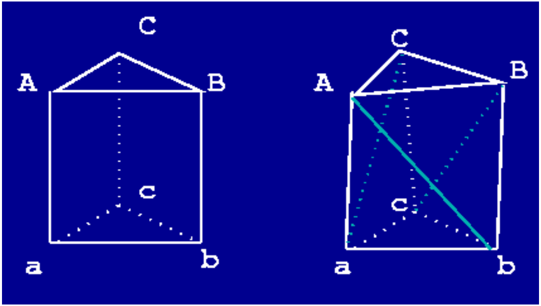
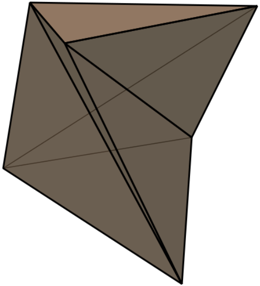
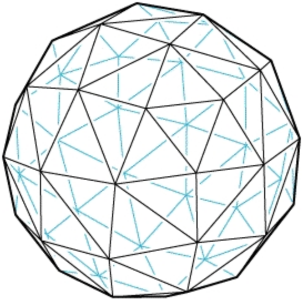
The Schönhardt polyhedron





Example 3 : the boundary complex of the convex hull of a finite set of points in general position



Polytope

$$\mathrm{conv}(\mathcal{P}) = \{x \in \mathbb{R}^d,\ x = \sum_{i=0}^{k} \lambda_i\, p_i,\ \lambda_i \in [0,1],\ \sum_{i=0}^{k} \lambda_i = 1\}$$

Supporting hyperplane H :
$H \cap \mathcal{P} \neq \emptyset$, $\mathcal{P}$ on one side of H

Faces : $\mathrm{conv}(\mathcal{P}) \cap H$, H supp. hyp.

- $\mathcal{P}$ is in general position iff no subset of $k+2$ points lie in a k-flat
- If P is in general position, all faces of $\mathrm{conv}(\mathcal{P})$ are simplices

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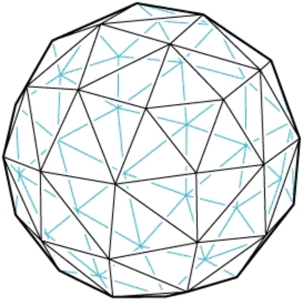

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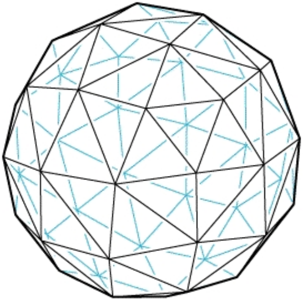

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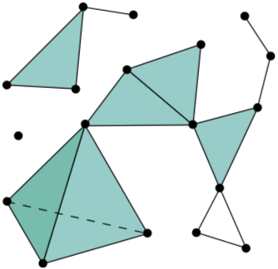
Abstract simplicial complexes



H. Poincaré (1854-1912)

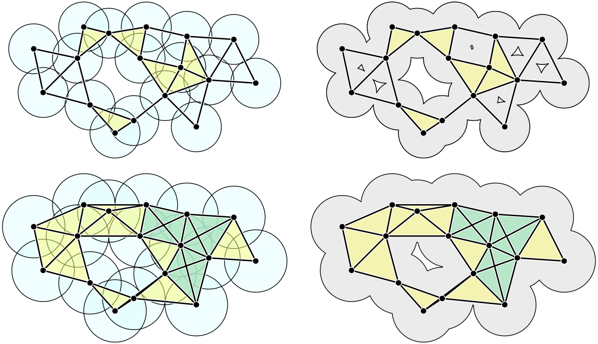
Let V be a finite set. A simplicial (abstract) complex on V is a finite set of subsets of V called the simplices or faces of K that satisfy :

1. The elements of V belong to K (vertices)
2. If $\tau \in K$ and $\sigma \subseteq \tau$, then $\sigma \in K$

The dimension of a complex is the maximum dimension of its simplices

Nerve of a finite cover $\mathcal{U} = \{U_1, ..., U_n\}$ of X

An example of an abstract simplicial complex

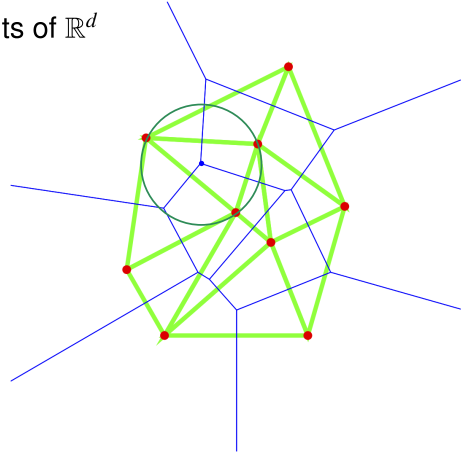


The nerve of $\mathcal{U}$ is the (abstract) simplicial complex $K(U)$ defined by

$$\sigma = [U_{i_0}, ..., U_{i_k}] \in K(U) \quad \Leftrightarrow \quad \cap_{i=1}^{k} U_{i_j} \neq \emptyset$$

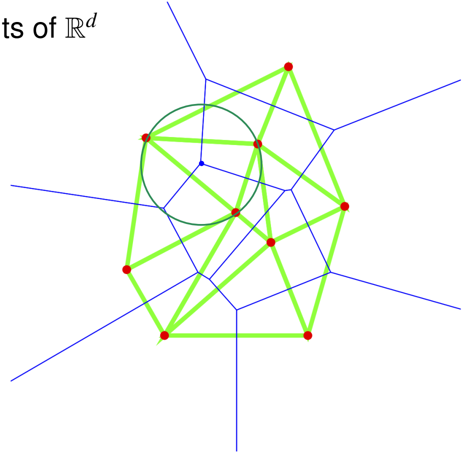
The Delaunay complex

$\mathcal{P}$ a finite set of points of $\mathbb{R}^d$



- The Delaunay complex $\mathrm{Del}(\mathcal{P})$ of $\mathcal{P}$ is the nerve of $\mathrm{Vor}(\mathcal{P})$
- Cannot be realized in $\mathbb{R}^d$ if $\mathcal{P}$ is not in general position wrt spheres

$\mathcal{P}$ a finite set of points of $\mathbb{R}^d$



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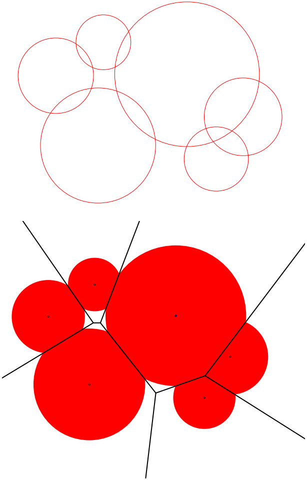
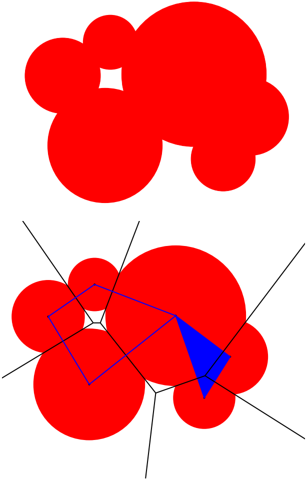
(Weighted) alpha-complex





The alpha-complex has the same homotopy type as the union of balls

Realization of an abstract simplicial complex

- A realization of an abstract simplicial complex K is a geometric simplicial complex K_g whose corresponding abstract simplicial complex is isomorphic to K, i.e.

$$\exists \text{ bijective } f : \text{vert}(K) \rightarrow \text{vert}(K_g) \quad \text{s.t.} \quad \sigma \in K \quad \Rightarrow \quad f(\sigma) \in K_g$$

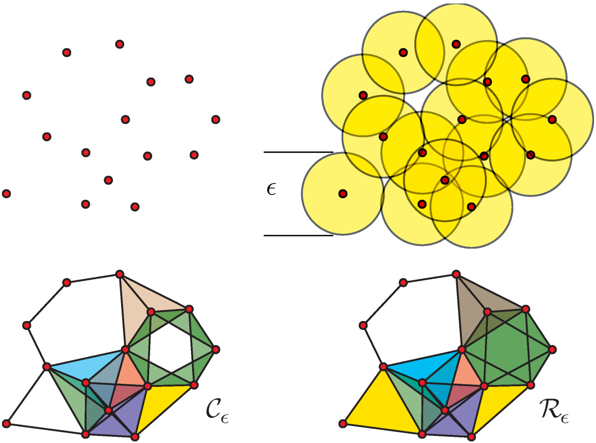
- Any abstract simplicial complex K can be realized in $\mathbb{R}^n$

  Hint : $v_i \rightarrow p_i = (0, ..., 0, 1, 0, ...0) \in \mathbb{R}^n$ $\quad (n = \sharp\text{vert}(K))$
  $\sigma = \text{conv}(p_1, ..., p_n)$ $\quad$ (canonical simplex)
  $K_g \subseteq \sigma$

- Realizations are not unique but are all topologically equivalent (homeomorphic)

Rips complex

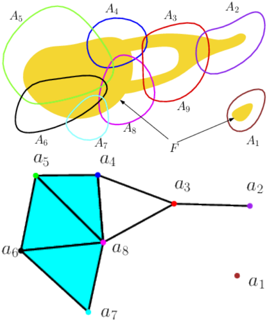
$$\sigma \subseteq \mathcal{P} \in R(\mathcal{P}, \alpha) \quad \Leftrightarrow \quad \forall p, q \in \sigma \ \|p - q\| \leq \alpha \quad \Leftrightarrow \quad B\left(p, \frac{\alpha}{2}\right) \cap B\left(q, \frac{\alpha}{2}\right) \neq \emptyset$$

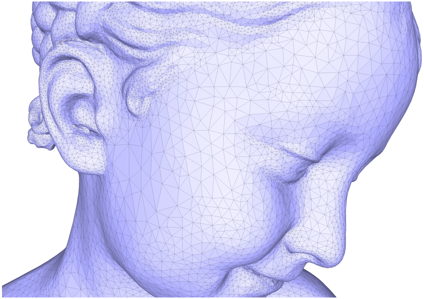


Construction of the Rips complex

- Interleaving : $R(\mathcal{P},\alpha) \subseteq C(\mathcal{P},\alpha) \subseteq R(\mathcal{P},2\alpha)$
- Computing $R(\mathcal{P},\alpha)$ reduces to computing the graph G (vertices+edges) of $R(\mathcal{P},\alpha)$ and the cliques of G

Triangulation of topological spaces





Triangulation of a topological space $\mathbb{X}$

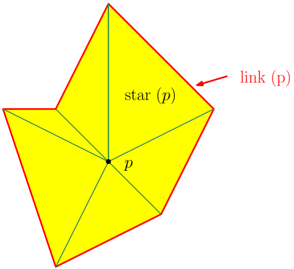
A simplicial complex homeomorphic to $\mathbb{X}$

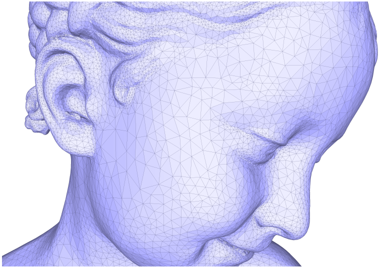
See the next lectures

Combinatorial (PL) manifolds

Definition

A simplicial complex $\hat{\mathcal{S}}$ is a PL manifold of dimension k iff the link of each vertex is the triangulation of a topological sphere of dimension k

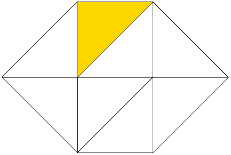




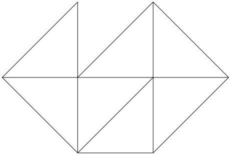
The underlying space of a PL manifold is a topological manifold

Data structures to represent simplicial complexes

Atomic operations



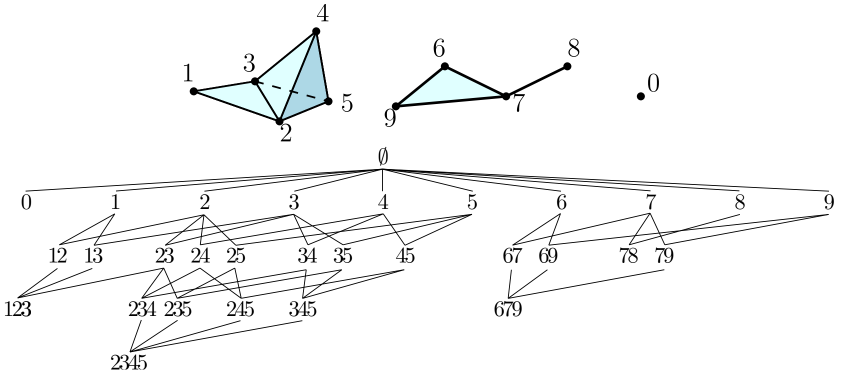
- **Look-up/Insertion/Deletion** of a simplex
- **Facets** and **subfaces** of a simplex
- **Cofaces**, **link** of a simplex
- **Topology preserving** operations
  - Edge contractions
  - Elementary collapses



Explicit representation of all simplices ? of all incidence relations ?

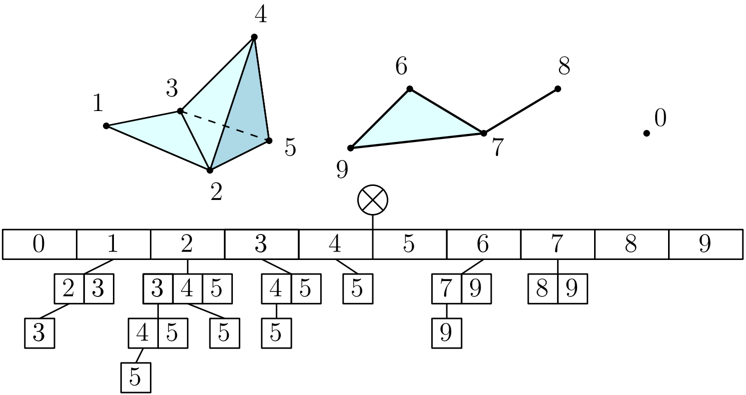
The Hasse diagram

$$G(V,E) \qquad \sigma \in V \quad \Leftrightarrow \quad \sigma \in K$$
$$(\sigma,\tau) \in E \quad \Leftrightarrow \quad \sigma \subset \tau \quad \wedge \quad \dim(\sigma) = \dim(\tau) - 1$$



The simplex tree is a prefix tree (trie)

1. index the vertices of K
2. associate to each simplex $\sigma \in K$, the sorted list of its vertices
3. store the simplices in a trie.



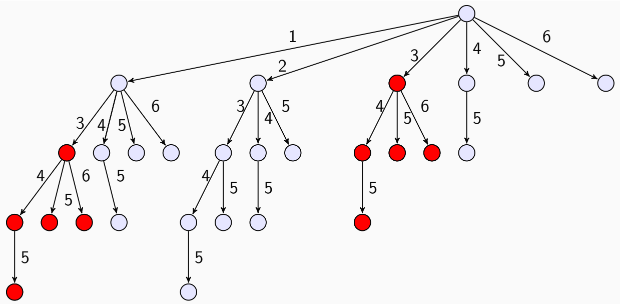
Performance of the simplex tree

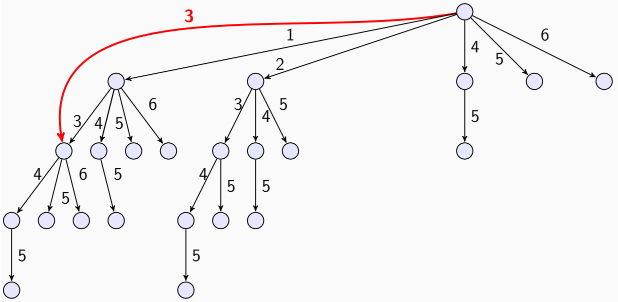
- Explicit representation of all simplices
- #nodes $= \#\mathcal{K}$
- depth $= \dim(\mathcal{K}) + 1$
- #children$(\sigma) \leq$ #cofaces$(\sigma) \leq$ deg(last(σ))
- **Memory complexity** : $O(1)$ per simplex
- **Basic operations**
  - Membership (σ) : $O(d_\sigma \log n)$
  - Insertion (σ) : $O(2^{d_\sigma} d_\sigma \log n)$

Data	$\|\mathcal{P}\|$	D	d	r	k	T_g	$\|E\|$	T_{Rips}	$\|\mathcal{K}\|$	T_{tot}	$T_{\mathrm{tot}}/\|\mathcal{K}\|$
Bud	49,990	3	2	0.11	3	1.5	1,275,930	104.5	354,695,000	104.6	$3.0 \cdot 10^{-7}$
Bro	15,000	25	?	0.019	25	0.6	3083	36.5	116,743,000	37.1	$3.2 \cdot 10^{-7}$
Cy8	6,040	24	2	0.4	24	0.11	76,657	4.5	13,379,500	4.61	$3.4 \cdot 10^{-7}$
Kl	90,000	5	2	0.075	5	0.46	1,120,000	68.1	233,557,000	68.5	$2.9 \cdot 10^{-7}$
S4	50,000	5	4	0.28	5	2.2	1,422,490	95.1	275,126,000	97.3	$3.6 \cdot 10^{-7}$

Implemented in the GUDHI library

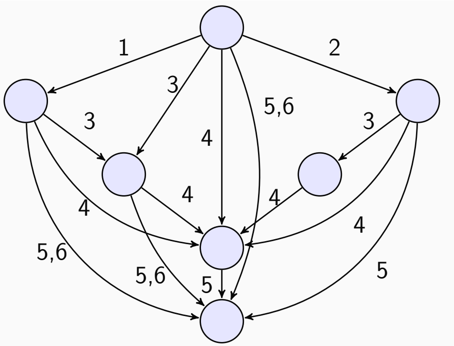
Redundancy in the Simplex Tree





Minimal simplex automaton

B., Karthik, Tavenas 2016



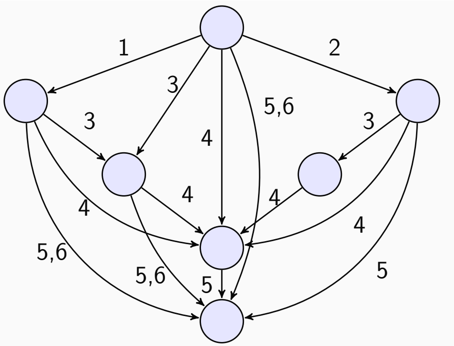
- Compression time : $O(m \log m \log n)$ [Hopcroft 1971]
- Static queries : unchanged
- Dynamic queries : more complex

- The size of the automaton depends on the labelling of the vertices

  Finding a minimal automaton is NP-complete

Minimal simplex automaton

B., Karthik, Tavenas 2016



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Experiments

Data Set 1 : Rips Complex from sampling of Klein bottle in $\mathbb{R}^5$.

n	α	d	k	m	Size After Compression	Compression Ratio
10,000	0.15	10	24,970	604,573	218,452	2.77
10,000	0.16	13	25,410	1,387,023	292,974	4.73
10,000	0.17	15	27,086	3,543,583	400,426	8.85
10,000	0.18	17	27,286	10,508,486	524,730	20.03

Data Set 2 : Flag complexes generated from random graph $G_{n,p}$.

n	p	d	k	m	Size After Compression	Compression Ratio
25	0.8	17	77	315,370	467	537.3
30	0.75	18	83	4,438,559	627	7,079.0
35	0.7	17	181	3,841,591	779	4,931.4
40	0.6	19	204	9,471,220	896	10,570.6
50	0.5	20	306	25,784,504	1,163	22,170.7

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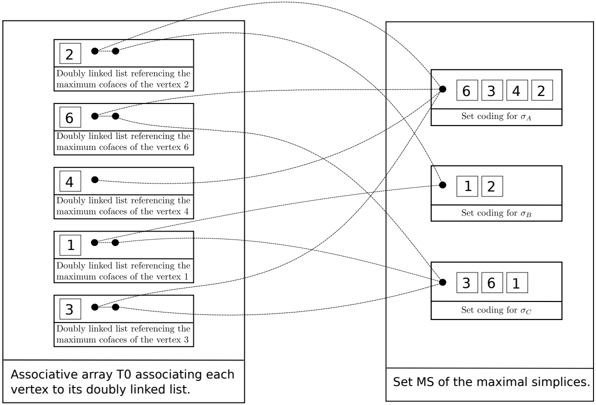
Simplex Array List

[B., Karthik C.S., Tavenas 2017]

Store only the maximal simplices



Memory storage : $O\left(\sum_{\sigma \in K} d_\sigma\right) = O(kd)$ Optimal

Proof of optimality

Theorem

Consider the class of all simplicial complexes $\mathcal{K}(n, k, d)$ where $d \geq 2$ and $k \geq n + 1$.

Any data structure that can represent the simplicial complexes of this class requires $\log \binom{\binom{n/2}{d+1}}{k-n}$ bits to be stored,

which is $\Omega(kd \log n)$ for any constant $\varepsilon \in (0, 1)$ and for $\frac{2}{\varepsilon} n \leq k \leq n^{(1-\varepsilon)d}$ and $d \leq n^{\varepsilon/3}$.

Proof $\mathcal{P} = |\text{vert}(K)|$, $\mathcal{P}' \subset \mathcal{P}$, $|\mathcal{P}'| = n/2$

Consider the set S of all simplicial complexes with vertex set $\subset \mathcal{P}'$, of dimension d and having $k - n$ maximal simplices (all of dimension d) and observe that $|S| = \binom{\binom{n/2}{d+1}}{k-n}$

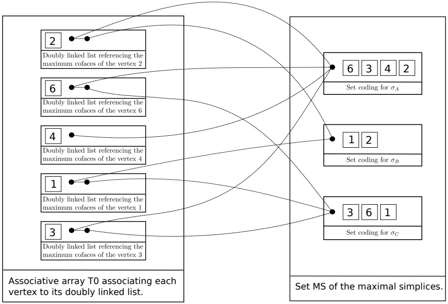
Let $K_1, ..., K_{|S|}$ be those complexes with vertex sets $\mathcal{P}_1, ..., \mathcal{P}_{|S|}$

Complete each K_i with vertices in $\mathcal{P} \setminus \mathcal{P}_i$ and edges spanning those vertices so that K_i^+ has n vertices and k maximal simplices (of dimension 1 or h)

We have $|S|$ complexes of $\mathcal{K}(n, k, d, m)$

Basic operations

Complexity depends on a local parameter



$\Gamma_i(\sigma) =$ number of maximal cofaces of σ of dimension i

$\Gamma_i = \max_{\sigma \in K} \Gamma_i(\sigma)$

Membership (σ) : $O\left(\sum_{i=0}^{d_\sigma - 1} \Gamma_i(\sigma)\right) = O(\Gamma_0 d \log n)$ ST : $O(d \log n)$

Insertion (σ) : $O(\Gamma_0(\sigma) d_\sigma^2 \log n) = O(\Gamma_0 d^2)$ ST : $O(d_\sigma 2^{d_\sigma} \log n)$

Experimental results

Data Set 1 (Rips complex on a Klein bottle in $\mathbb{R}^5$)

No	n	α	d	k	m	Γ_0	Γ_1	Γ_2	Γ_3	$\|SAL\|$
1	10,000	0.15	10	24,970	604,573	62	53	47	37	424,440
2	10,000	0.16	13	25,410	1,387,023	71	61	55	48	623,238
3	10,000	0.17	15	27,086	3,543,583	90	67	61	51	968,766
4	10,000	0.18	17	27,286	10,508,486	115	91	68	54	1,412,310

To be released in the GUDHI library (F. Godi)

Conclusions

Next lectures

- Other types of simplicial complexes
- Triangulation of manifolds

Open questions

- Bound on Γ_0 for interesting simplicial complexes
- Lower bounds on query time assuming optimal storage $O(kd \log n)$