Delaunay Triangulations of Manifolds

5. Delaunay triangulation of manifolds

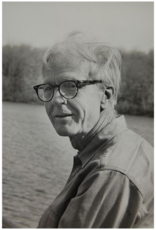
Jean-Daniel Boissonnat

Winter School on Computational Geometry
Tehran Polytechnic
February 28 - March 5, 2018

Delaunay Triangulations of Manifolds

Triangulation of manifolds

A long standing problem



Hassler Whitney

- A question of Poincaré
- Positive answer for differentiable manifolds
  [Cairns, Whitney, Whitehead 1940-50]
- Positive answer for topological manifolds of dimension ≤ 3
  [Moïse 1952]
- Negative answer for topological manifolds of dimension ≥ 4
  [Manolescu 2013]
- Last lecture : reconstruction of smooth submanifolds of $\mathbb{R}^d$

  Today : meshing algorithms, Riemannian (intrinsic) manifolds

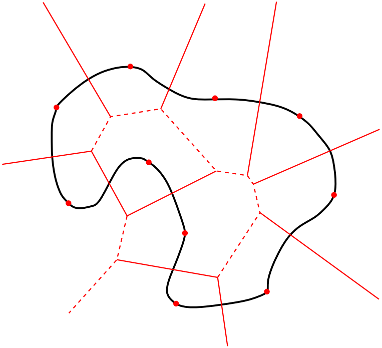
1. Surface mesh generation

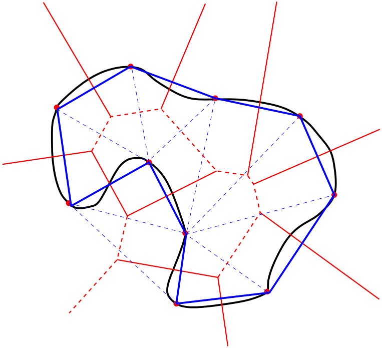
2. Anisotropic Delaunay triangulations

3. Delaunay triangulation of Riemannian manifolds

Restricted Delaunay triangulation $\mathrm{Del}_{|S}(\mathcal{P})$

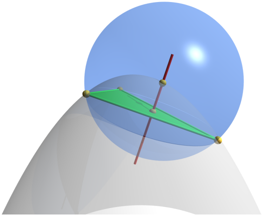
[Chew 93]





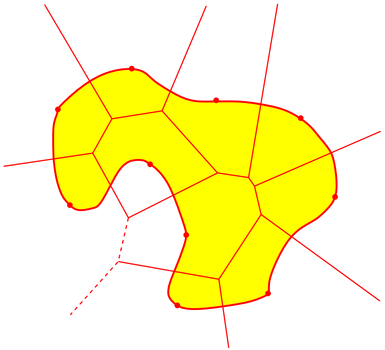
$$\mathrm{Del}_{|\mathcal{S}}(\mathcal{P}) = \{f \in \mathrm{Del}(\mathcal{P}) \mid \mathrm{Vor}(f) \cap \mathcal{S} \neq \emptyset\}$$

$= \{f \in \mathrm{Del}(\mathcal{P}) \mid \exists$ empty ball B_f
circumscribing f and centered on $\mathcal{S}\}$



Meshing volumes bounded by surfaces

Restricted Delaunay triangulation $\mathrm{Del}_{|\Omega}(\mathcal{P})$



$$\mathrm{Del}_{|\mathcal{S}}(\mathcal{P}) = \{f \in \mathrm{Del}(\mathcal{P}) \mid \mathrm{Vor}(f) \cap \Omega \neq \emptyset\}$$

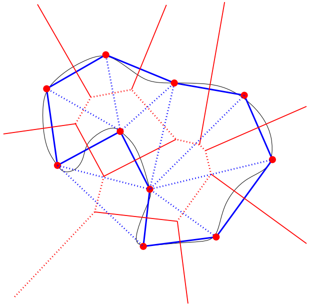
$= \{f \in \mathrm{Del}(\mathcal{P}) \mid \exists$ empty ball B_f circumscribing f and centered in $\Omega\}$

A variant of the nerve theorem [Edelsbrunner & Shah 1997]

Let $\mathbb{M}$ be a compact manifold without boundary. If, for any face $f \in \mathrm{Vor}(\mathsf{P})$ s.t. $f \cap \mathbb{M} \neq \emptyset$,

1. f intersects $\mathbb{M}$ transversally
2. $f \cap \mathbb{M} = \emptyset$ or is a topological ball

then $\mathrm{Del}_{\mathbb{M}}(\mathsf{P}) \approx \mathbb{M}$

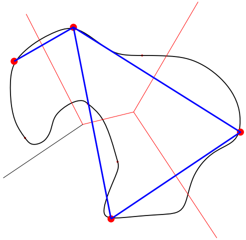


A variant of the nerve theorem [Edelsbrunner & Shah 1997]

Let $\mathbb{M}$ be a compact manifold without boundary. If, for any face $f \in \mathrm{Vor}(\mathsf{P})$ s.t. $f \cap \mathbb{M} \neq \emptyset$,

1. f intersects $\mathbb{M}$ transversally
2. $f \cap \mathbb{M} = \emptyset$ or is a topological ball

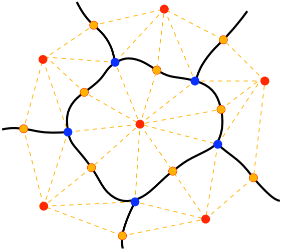
then $\mathrm{Del}_{\mathbb{M}}(\mathsf{P}) \approx \mathbb{M}$



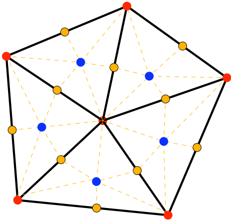
Proof of the closed ball property

Barycentric subdivision

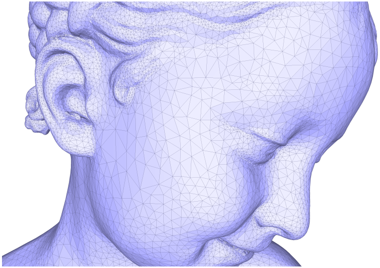
of $\mathrm{Vor}(\mathsf{P}) \cap \mathbb{M}$

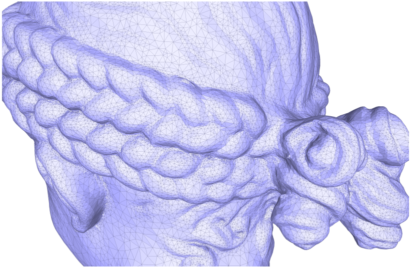


of $\mathrm{Del}_{\mathbb{M}}(\mathsf{P})$



Approximation theorem for surfaces





Amenta et Bern [1998], Amenta et Dey [2002]
Boissonnat et Cazals [2000], Boissonnat et Oudot [2005]

If $\mathcal{S}$ is a surface of $\mathbb{R}^3$ that is compact, without boundary and of positive reach $\mathrm{rch}(\mathcal{S}) > 0$

if $\mathcal{P}$ is an ε-net of $\mathcal{S}$ for a sufficiently small ε

alors $\mathrm{Del}_{|\mathcal{S}}(\mathcal{P})$ is a triangulation of $\mathcal{S}$ with all good properties

Surface mesh generation

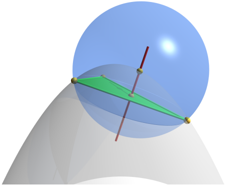
Delaunay refinement

[Chew 1993, B. & Oudot 2003]

$\phi : S \to \mathbb{R}$ = Lipschitz function
$\forall x \in S,\ 0 < \phi_0 \leq \phi(x) < \varepsilon\,\mathrm{lfs}(x)$

ORACLE : For a facet f of $\mathrm{Del}_{|S}(\mathcal{P})$,
return c_f, r_f and $\phi(c_f)$

A facet f is bad if $r_f > \phi(c_f)$



Algorithm

INIT Take a (small) initial sample $\mathcal{P}_0 \subset S$

REPEAT IF f is a bad facet
insert_in_Del3D(c_f) ,
update $\mathcal{P}$ et $\mathrm{Del}_{|S}(\mathcal{P})$

UNTIL there is no more bad facets

Surface mesh generation

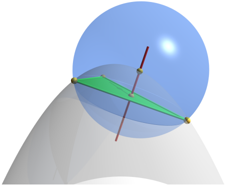
Delaunay refinement

[Chew 1993, B. & Oudot 2003]

$\phi : S \to \mathbb{R}$ = Lipschitz function
$\forall x \in S,\ 0 < \phi_0 \le \phi(x) < \varepsilon\,\mathrm{lfs}(x)$

ORACLE : For a facet f of $\mathrm{Del}_{|S}(\mathcal{P})$,
return c_f, r_f and $\phi(c_f)$

A facet f is bad if $r_f > \phi(c_f)$



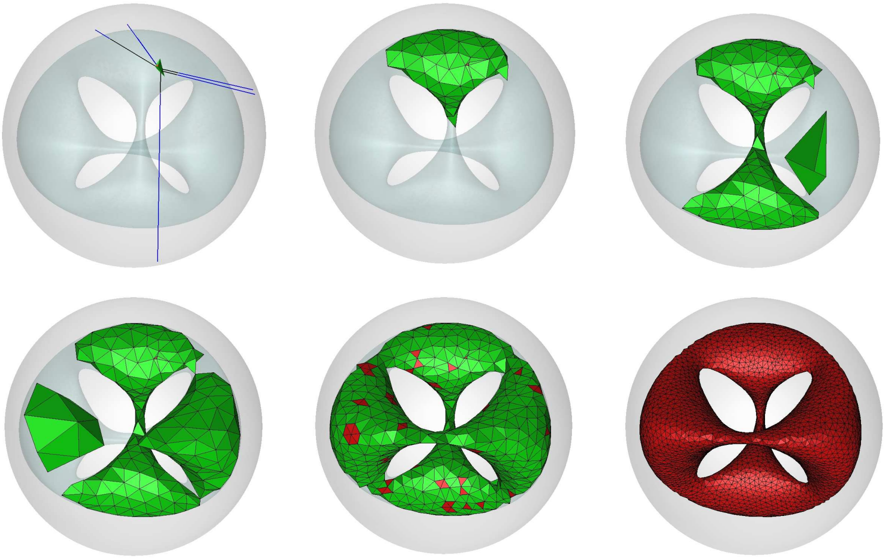
Algorithm

INIT Take a (small) initial sample $\mathcal{P}_0 \subset S$

REPEAT IF f is a bad facet
insert_in_Del3D(c_f) ,
update $\mathcal{P}$ et $\mathrm{Del}_{|S}(\mathcal{P})$

UNTIL there is no more bad facets

The algorithm in action



Properties of the meshing algorithm

Theorem

If $\mathcal{S}$ is a surface with positive reach, the Delaunay refinement algorithm outputs

- an $(\varepsilon + O(\varepsilon^2))$-net $\mathcal{P}$ of $\mathcal{S}$
- a PL surface $\hat{S}$
  - homeomorphic to S
  - close to $\mathcal{S}$

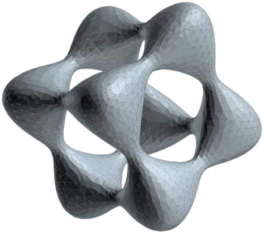
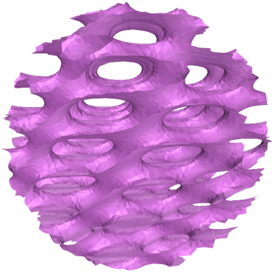
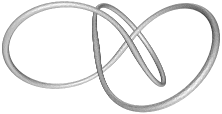
    Hausdorff and Fréchet distances $= O(\varepsilon^2)$

    normal approximation $= O(\varepsilon)$
  - the facets are thick

The oracle allows to mesh surfaces represented under various

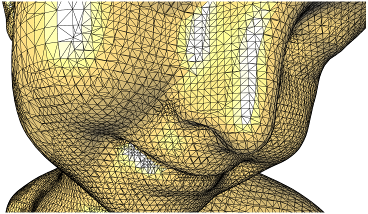
- Implicit surfaces $f(x, y, z) = 0$
- Isosurfaces in 3d images (Medical images)
- Remeshing of triangulated surfaces
- Surface reconstruction

Implicit surfaces

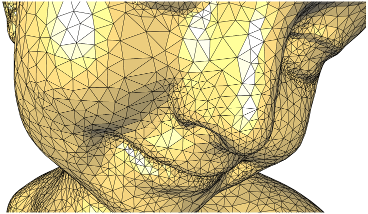




Remeshing of triangulated surfaces

Lipschitz surfaces [B., Oudot 2006]



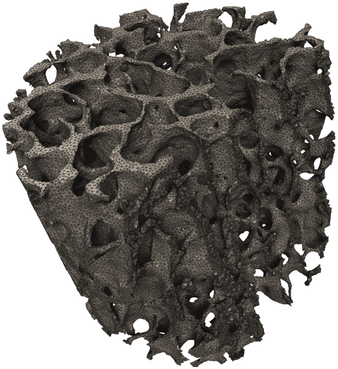
Marching cube

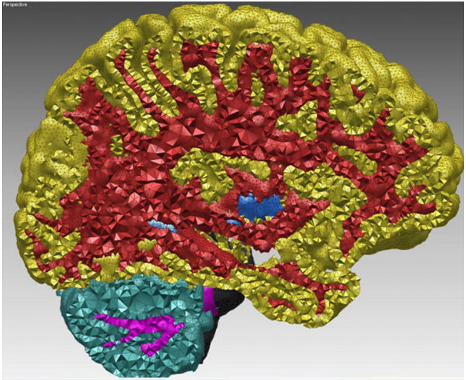


Delaunay refinement

Mesh gesh generation of 3D volumes

Delaunay refinement in higher dimensions



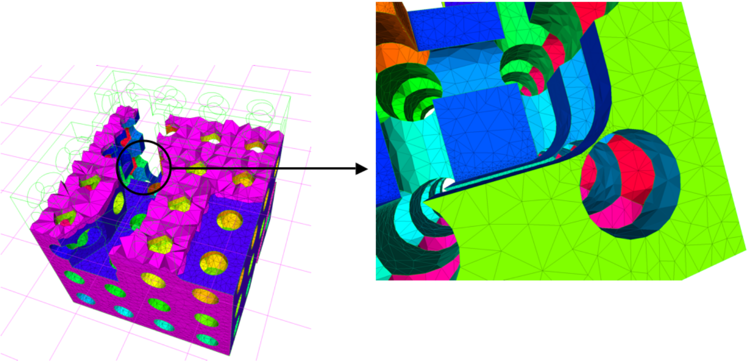


Thick triangulations : see Lecture 2

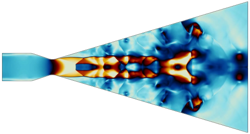
Sharp edges

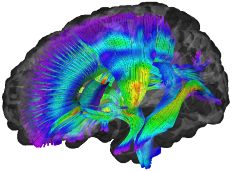
Use weighted Delaunay triangulations to conform to sharp edges

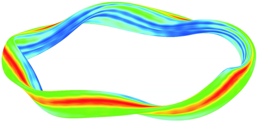
[Dey, Levine 2009], [Oudot, Rineau, Yvinec 2005]



Anisotropic mesh generation







Anisotropic mesh

- A simplicial complex
- Simplices are elongated along specified directions

Applications

- Mesh adaptation (solving PDE related to aniso. phenomena)
- Approximation of surfaces (curvature)
- Approximation of functions (Hessian)

Riemannian metric

Metric field G : Continuous map $G : x \in \Omega \mapsto G_x$
G_x positive symmetric definite matrix

Distances and lengths

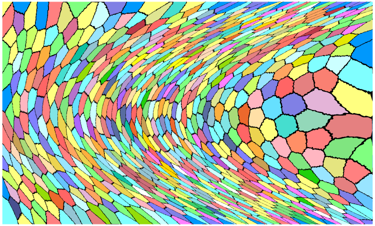
$$\mathrm{length}(\gamma) = \int \langle \dot\gamma, \dot\gamma \rangle_g^{1/2} \mathrm{d}t = \int \sqrt{\dot\gamma^t(t) G_{\gamma(t)} \dot\gamma(t)} \mathrm{d}t$$
$$d_G(p, q) = \inf_{\gamma} \mathrm{length}(\gamma)$$

Special cases

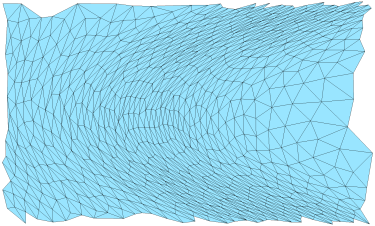
- Euclidean metric : $G_x = I$ at any x
- Uniform metric : $G_x = G$ at any x
- Approximation of functions defined over $\mathbb{R}^d$
  The anisotropy may be given by the Hessian

Riemannian Voronoi diagrams in $(\mathbb{R}^d, G)$

$$V_G(p_i) = \{x \in \mathbb{R}^d \mid d_G(p_i, x) \leq d_G(p_j, x), \forall p_j \in \mathcal{P} \setminus p_i\}.$$



Riemannian VD

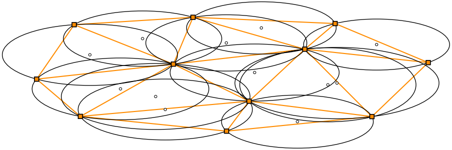


Nerve of the diagram

The Delaunay complex $\mathrm{Del}_G(\mathcal{P})$ is the nerve of $V_G(\mathcal{P})$

A simple case : uniform metric

$$\forall x,\ \ G_x = G_p$$
$$d_{G_p}(x,y) = \sqrt{(x-y)^t\, G_p\, (x-y)}$$



The associated Voronoi diagram $\mathrm{Vor}_{G_p}(\mathcal{P})$ is affine

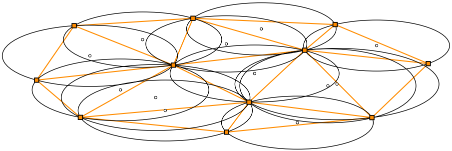
$$\begin{aligned} d_{G_p}(x,a) < d_{G_p}(x,b) \quad &\Leftrightarrow \quad (x-a)^t G_p (x-a) < (x-b)^t G_p (x-b) \\ &\Leftrightarrow \quad -2a^t G_p x + a^t G_p a < -2b^t G_p x + b^t G_p b \end{aligned}$$

Corollaries

+ The Delaunay complex $\mathrm{Del}_{G_p}(\mathcal{P})$ is an embedded triangulation if $\mathcal{P}$ is in general position
+ $\mathrm{Del}_{G_p}(\mathcal{P})$ can be computed efficiently

A simple case : uniform metric

$$\forall x,\ \ G_x = G_p$$
$$d_{G_p}(x,y) = \sqrt{(x-y)^t\, G_p\, (x-y)}$$

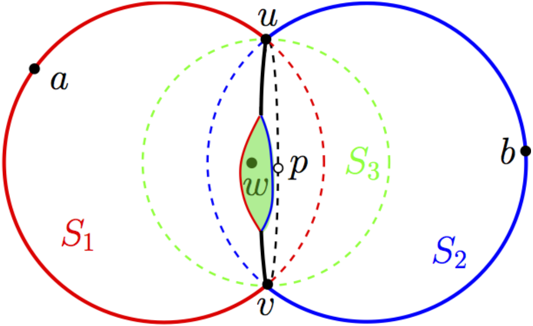


The associated Voronoi diagram $\mathrm{Vor}_{G_p}(\mathcal{P})$ is affine

$$\begin{aligned} d_{G_p}(x,a) < d_{G_p}(x,b) \quad &\Leftrightarrow \quad (x-a)^t G_p (x-a) < (x-b)^t G_p (x-b) \\ &\Leftrightarrow \quad -2a^t G_p x + a^t G_p a < -2b^t G_p x + b^t G_p b \end{aligned}$$

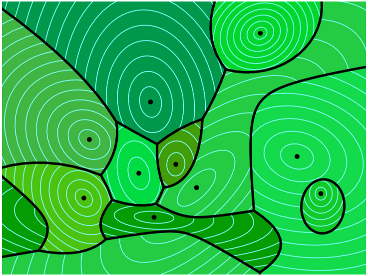
Corollaries

+ The Delaunay complex $\mathrm{Del}_{G_p}(\mathcal{P})$ is an embedded triangulation if $\mathcal{P}$ is in general position
+ $\mathrm{Del}_{G_p}(\mathcal{P})$ can be computed efficiently



Sampling density is not enough

- $V(p) = \{x : d_p(x,p) \leq d_q(x,q)$ for all $p,q \in P\}$ [Labelle & Shewchuk 2003]

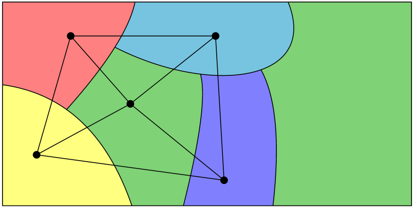


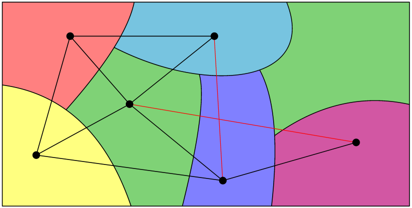
- $V(p) = \{x : d_x(x,p) \leq d_x(x,q)$ for all $p,q \in P\}$

[Du & Wang 2005] [Canas & Gortler 2012]

Some properties of anisotropic Voronoi diagrams

- LS-VAD is identical to the vertical projection of a Laguerre diagram in $\mathbb{R}^D$ to a quadratic d-manifold, $D = d(d+3)/2$
- Each site is within its cell
- Possibility of orphans (non-connected cells)
- The nerve may not be embedded



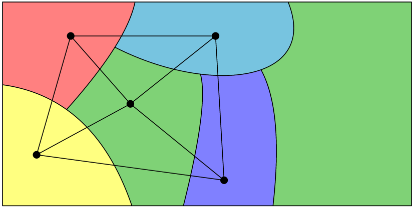


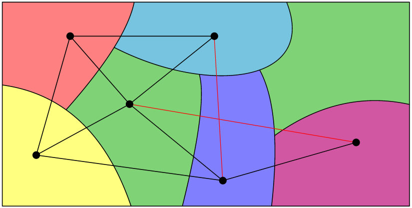
Guarantees

Termination and quality bounds proven in 2D [Labelle, Shewchuk '03] and surfaces [Cheng et al. '06], but no extension to higher dimensions

Some properties of anisotropic Voronoi diagrams

- LS-VAD is identical to the vertical projection of a Laguerre diagram in $\mathbb{R}^D$ to a quadratic d-manifold, $D = d(d+3)/2$
- Each site is within its cell
- Possibility of orphans (non-connected cells)
- The nerve may not be embedded



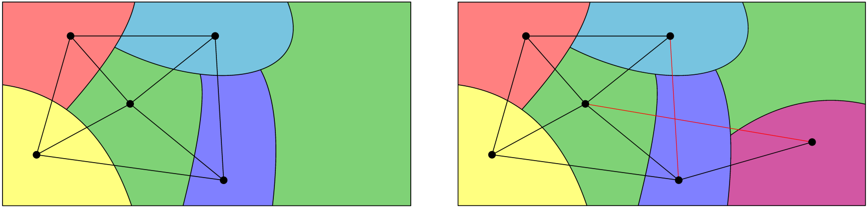


Guarantees

Termination and quality bounds proven in 2D [Labelle, Shewchuk '03] and surfaces [Cheng et al. '06], but no extension to higher dimensions

Some properties of anisotropic Voronoi diagrams

- LS-VAD is identical to the vertical projection of a Laguerre diagram in $\mathbb{R}^D$ to a quadratic d-manifold, $D = d(d+3)/2$
- Each site is within its cell
- Possibility of orphans (non-connected cells)
- The nerve may not be embedded

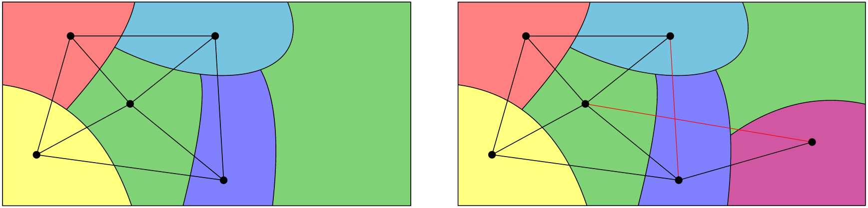


Guarantees

Termination and quality bounds proven in 2D [Labelle, Shewchuk '03] and surfaces [Cheng et al. '06], but no extension to higher dimensions

Some properties of anisotropic Voronoi diagrams

- LS-VAD is identical to the vertical projection of a Laguerre diagram in $\mathbb{R}^D$ to a quadratic d-manifold, $D = d(d+3)/2$
- Each site is within its cell
- Possibility of orphans (non-connected cells)
- The nerve may not be embedded



Guarantees

Termination and quality bounds proven in 2D [Labelle, Shewchuk '03] and surfaces [Cheng et al. '06], but no extension to higher dimensions

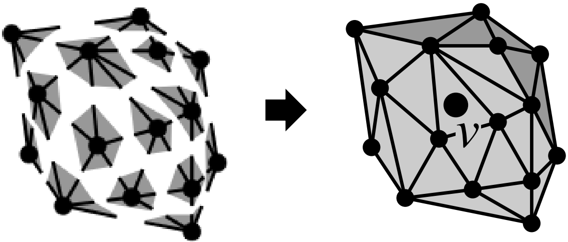
A dual approximation

Locally uniform Delaunay complex [B., Wormser, Yvinec 2015]

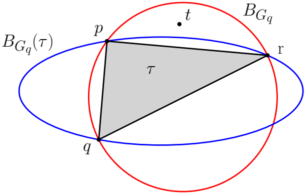
Definition A locally uniform Delaunay complex is a simplicial complex in which the star of each vertex is Delaunay for the (uniform) metric attached to the vertex

$$\forall p \in \mathcal{P}, \quad \mathrm{Del}_{G_{\mathcal{P}}}(\mathcal{P}) = \bigcup_{p \in \mathcal{P}} \mathrm{star}(p, \mathrm{Del}_{G_p}(\mathcal{P}))$$

Star stitching : make stars consistent s.t. $\mathrm{Del}_{G_{\mathcal{P}}}(\mathcal{P})$ is a global triangulation



Conflicting stars



$$\tau \in \mathrm{star}(p) \Rightarrow t \in B_{G_p}(\tau)$$
$$\tau \notin \mathrm{star}(p) \Rightarrow t \notin B_{G_q}(\tau)$$

If τ is small and fat, and the metric field is Lipschitz continuous

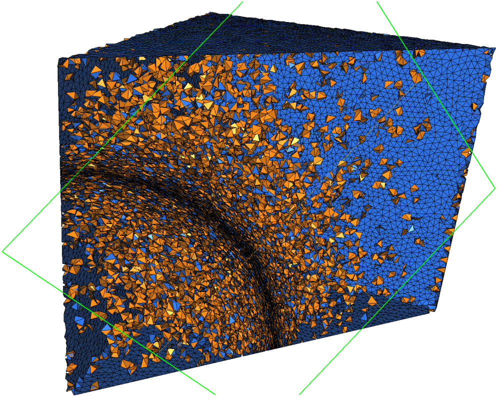
$\Rightarrow$ the vertices of the $(d+1)$-simplex $\tau * t$ are close to a $(d-1)$-sphere

Theorem : If $\mathcal{P}$ is a net which is sufficiently dense and protected, then all stars can be made consistent by perturbation and $\mathrm{Del}_{G_{\mathcal{P}}}(\mathcal{P})$ is an embedded triangulation

Extensive empirical study

M. Rouxel-Labbé 2016

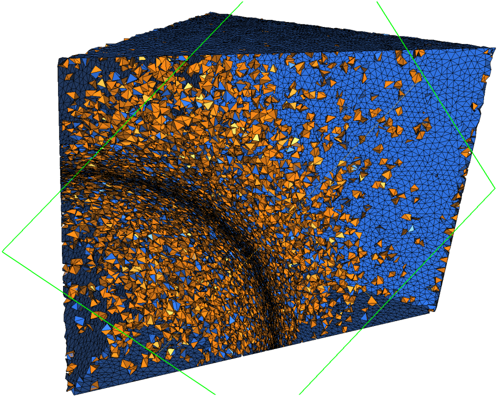
- Provably correct
- Extremely robust and (relatively) fast
- Elements conform to the anisotropy ...
- ...but not practical
  - No control over the number of elements
  - Exceedingly large amounts of vertices are generally required to achieve consistency



Extensive empirical study

M. Rouxel-Labbé 2016

- Provably correct
- Extremely robust and (relatively) fast
- Elements conform to the anisotropy ...
- ...but not practical
  - No control over the number of elements
  - Exceedingly large amounts of vertices are generally required to achieve consistency



1. Surface mesh generation

2. Anisotropic Delaunay triangulations

3. Delaunay triangulation of Riemannian manifolds

Riemannian manifolds

A manifold endowed with a metric while we ignore the ambient space

Definition : a (smooth) Riemannian manifold $(\mathbb{M}, G)$ is a real, smooth manifold $\mathbb{M}$ equipped with an inner product G_x on the tangent space T_x at each point x that varies smoothly from point to point

The family G_x of inner products is called a Riemannian metric (tensor)

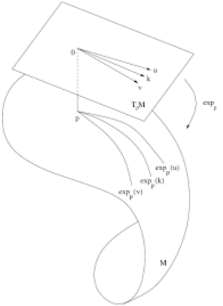
Exponential map

$\mathbb{M}$ a differentiable manifold and p a point of $\mathbb{M}$

$v \in T_p$ be a tangent vector to the manifold at p

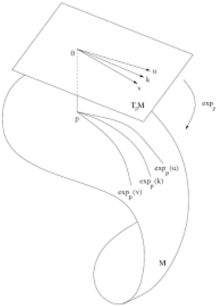
An affine connection on $\mathbb{M}$ allows one to define the notion of a geodesic through p

There is a unique geodesic γ_v satisfying $\gamma_v(0) = p$ with initial tangent vector $\gamma_v'(0) = v$



The corresponding exponential map is defined by

$$\exp_p(v) = \gamma_v(1)$$



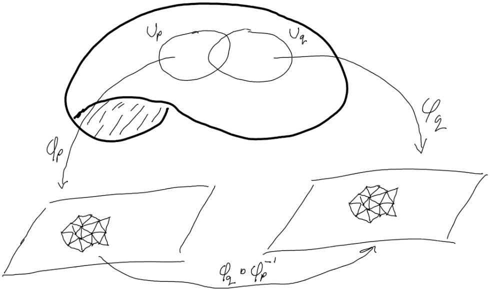
Theorem $\forall x, y \in B(p, r)$,

$$\left(1 - \frac{\Lambda r^2}{2}\right) d_G(x, y) \leq \| \exp_p(x) - \exp_p(y) \| \leq \left(1 + \frac{\Lambda r^2}{2}\right) d_G(x, y)$$

where Λ is a bound on the absolute value of the sectional curvature of $\mathbb{M}$

Manifold Delaunay complex

Euclidean charts and local triangulations [B., Dyer, Ghosh 2017]



- use the local Euclidean metric
- the metric is close to that on the manifold (Rauch th.)
- obtain protection in local coordinate charts
- the local stars are then consistent

Manifold Delaunay complex

(B., Dyer, Ghosh 2017)

$F \colon (X, d_X) \to (Y, d_Y)$ is a ξ-*distortion map* if

$$|d_Y(F(x), F(y)) - d_X(x, y)| \leq \xi d_X(x, y).$$

Definition ((ϵ, η_0)-net)

- ϵ a sampling radius (for each $x \in \mathbb{M}$, $d_G(x, \mathsf{P}) < \epsilon$)
- for each $p, q \in \mathsf{P}$, $d_G(p, q) \geq \eta_0 \epsilon$

Theorem (manifold Delaunay complex via perturbation)

- $\mathsf{P} \subset \mathbb{M}$ *a* (ϵ, η_0) *net in each coordinate chart*
- ϵ *a local sampling radius*
- *each* ϕ_p *is a* ξ*-distortion map,* $\xi \sim (\eta_0/2)^{m^3} \rho_0^m$*,*
- $\rho_0 = \rho/\epsilon < \eta_0/4$ *bounds the magnitude of the perturbation* ρ

Then the perturbation algorithm produces a manifold Delaunay complex $\mathrm{Del}(\mathsf{P}')$ *for* $\mathbb{M}$*.*

Riemannian Delaunay triangulation

(Dyer, Vegter, Wintraecken, 2015) ; (B., Dyer, Ghosh 2017)

Theorem (Riemannian DT)

If $\mathsf{P} \subset \mathbb{M}$ *is a* (ϵ, η_0)*-net with*

$$\epsilon \leq \min\{\frac{1}{4}\iota_{\mathbb{M}},\ \sim \Lambda^{-\frac{1}{2}}(\eta_0/2)^{m^3}\rho_0^m\},$$

then

- $\mathrm{Del}(\mathsf{P}')$ *is a Delaunay triangulation*
- *it admits a piecewise flat metric defined by geodesic edge lengths*
- *the barycentric coordinate map* $H : |\mathrm{Del}(\mathsf{P}')| \to \mathbb{M}$ *is a* ξ*-distortion map with* $\xi \sim (\eta_0/2)^{m^3}\rho_0^m\Lambda\epsilon^2$ (they're Gromov–Hausdorff close)

Local metric criteria for triangulation

B., Dyer, Ghosh, Wintraecken 2018

Theorem (triangulation)

$H \to |\mathcal{A}| \to \mathbb{M}$ is a homeomorphism if we have (for all $p \in \mathsf{P}$) :

1. **compatible atlases**
2. **simplex quality** *Every simplex $\boldsymbol{\sigma} \in \mathrm{star}(p) = \widehat{\Phi}_p(\mathrm{star}(p))$ satisfies $s_0 \leq L(\boldsymbol{\sigma}) \leq L_0$ and $t(\boldsymbol{\sigma}) \geq t_0$.*
3. **distortion control** *$F_p = \phi_p \circ H \circ \widehat{\Phi}_p^{-1} \to |\mathrm{star}(p)| \to \mathbb{R}^m$, when restricted to any m-simplex in $\mathrm{star}(p)$, is an orientation-preserving ξ-distortion map with*

$$\xi < \frac{s_0 t_0^2}{12 L_0} = \frac{1}{12}\mu_0 t_0^2.$$

4. **vertex sanity** *For all other vertices $q \in \mathsf{P}$, if $\phi_p \circ H(q) \in |\mathrm{star}(p)|$, then q is a vertex of $\mathrm{star}(p)$.*

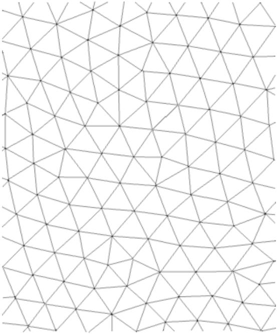
Geodesic distances cannot be computed exactly $\Rightarrow$ must discretize

Canvas

The underlying structure used to compute geodesic distances

Many methods exist to compute geodesic distances and paths :

- Fast marching methods [Konukoglu et al. '07]
- Heat-kernel based methods [Crane et al. '13]
- Short-term vector Dijkstra [Campen et al. '13]



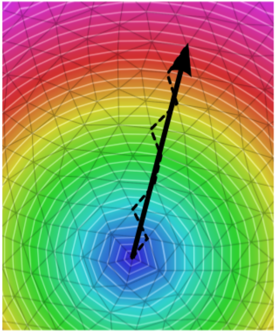
Geodesic distances cannot be computed exactly ⇒ must discretize

Canvas

The underlying structure used to compute geodesic distances

Many methods exist to compute geodesic distances and paths :

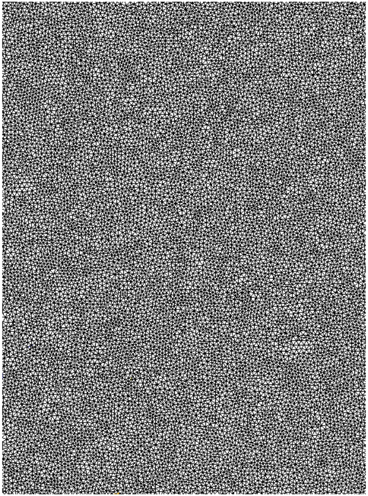
- Fast marching methods [Konukoglu et al. '07]
- Heat-kernel based methods [Crane et al. '13]
- Short-term vector Dijkstra [Campen et al. '13]



Construction of the discrete diagram

- Generate the canvas
- Color each vertex of the canvas with the closest site
- Farthest point refinement algorithm for new sites
- Extract the nerve

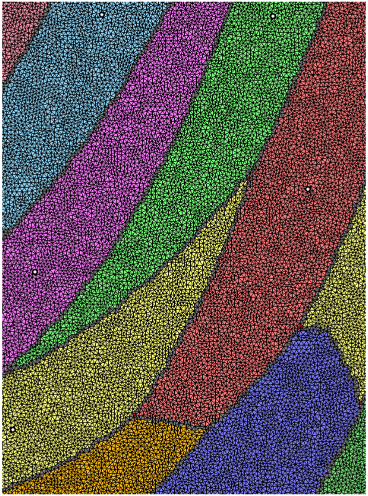
The discrete Delaunay complex is abstract, we define *straight* and *curved* realizations



Construction of the discrete diagram

- Generate the canvas
- Color each vertex of the canvas with the closest site
- Farthest point refinement algorithm for new sites
- Extract the nerve

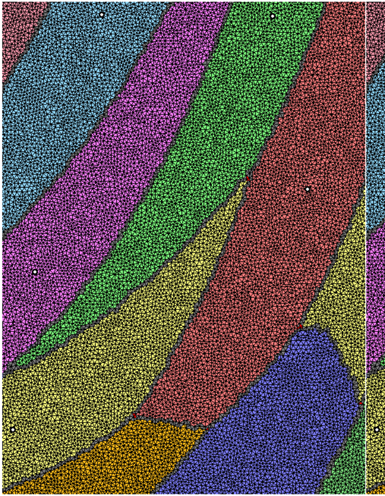
The discrete Delaunay complex is abstract, we define *straight* and *curved* realizations



Construction of the discrete diagram

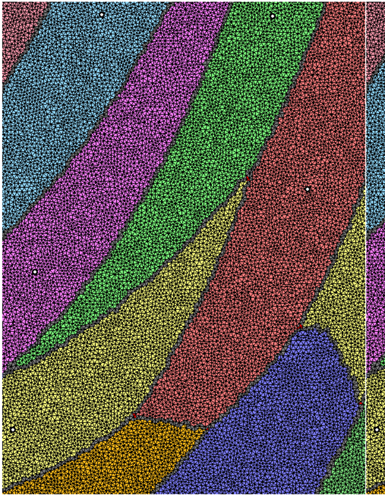
- Generate the canvas
- Color each vertex of the canvas with the closest site
- Farthest point refinement algorithm for new sites
- Extract the nerve

The discrete Delaunay complex is abstract, we define *straight* and *curved* realizations



Construction of the discrete diagram

- Generate the canvas
- Color each vertex of the canvas with the closest site
- Farthest point refinement algorithm for new sites
- Extract the nerve

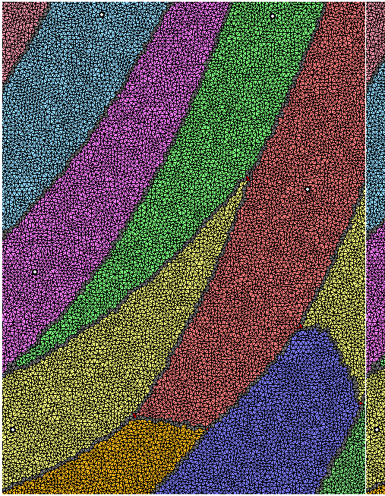


The discrete Delaunay complex is abstract, we define *straight* and *curved* realizations

Construction of the discrete diagram

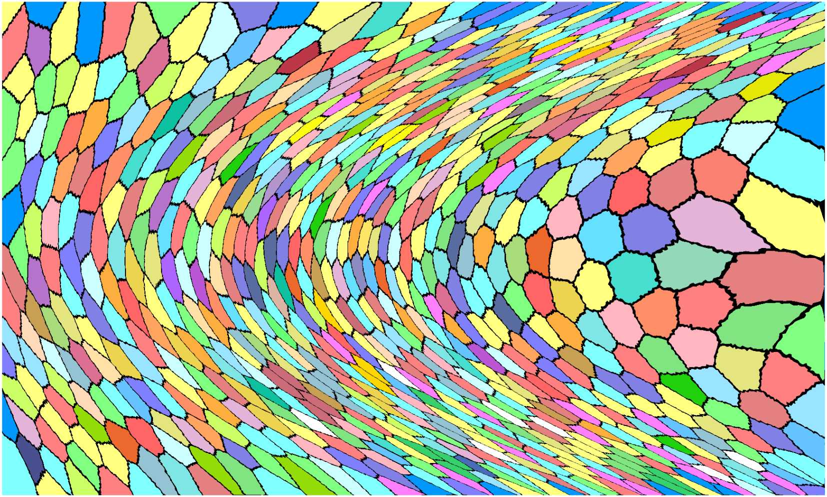
- Generate the canvas
- Color each vertex of the canvas with the closest site
- Farthest point refinement algorithm for new sites
- Extract the nerve

The discrete Delaunay complex is abstract, we define *straight* and *curved* realizations



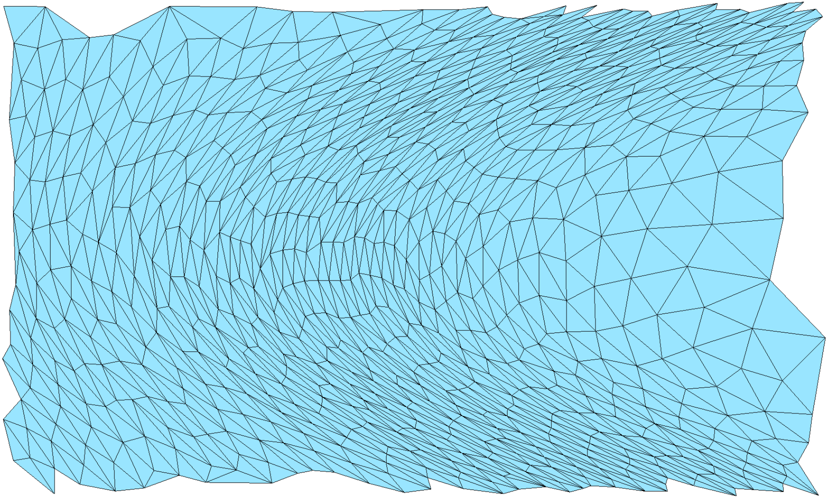
Discrete Riemannian Voronoi diagram

An example with 750 sites



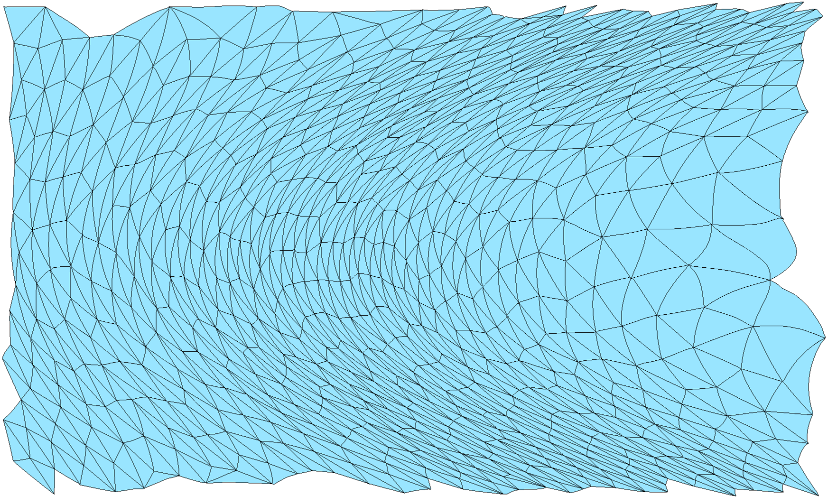
Straight Delaunay triangulation

Delaunay complex realized using straight edges



Curved Delaunay triangulation

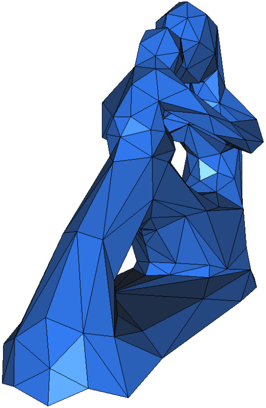
Delaunay complex realized using curved edges



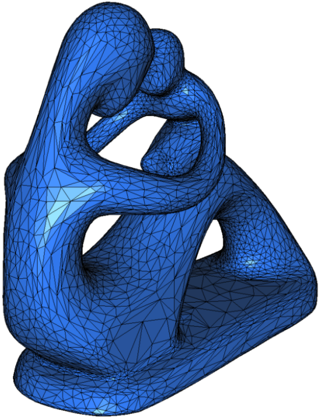
Riemannian Delaunay triangulation

Results

[B., Rouxel-Labbé, Wintraecken 2017]



220 sites



6027 sites

Delaunay Triangulations of Manifolds

A summary of the course

1. Delaunay triangulations in Euclidean and Laguerre geometry
   - Affine diagrams, first attempts to go beyond the Euclidean case
2. Good triangulations and meshes
   - Nets, protection, stability of combinatorial structures, randomized algorithms, LLL
3. Triangulation of topological spaces
   - Simplicial complexes for geometry modelling in higher dimensions
4. Shape reconstruction
   - Submanifolds, curse of dimensionality, intrinsic dimension
5. Delaunay triangulation of manifolds
   - Local (Riemannian) metric, anisotropic meshes, protection and stability again

Bibliography

H. Edelsbunner
Geometry and Topology for Mesh Generation, Cambridge 2001

T. Dey
Curve and surface reconstruction, Cambridge 2006

H. Edelsbrunner and J. Harer
Computational Topology, AMS 2010

S-W. Cheng, T. Dey, J. Shewchuk
Delaunay Mesh Generation, CRC Press 2012.

J-D. Boissonnat, F. Chazal, M. Yvinec
Geometric and Topological Inference, Cambridge 2018