Algorithms for Comparing Curves and Surfaces

3: Lower Bounds

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Computational Complexity

RUB

for two polygonal curves of complexity n each:

- Alt and Godau, 1992/95: $O(n^2 \log n)$ time algorithm for the Fréchet distance
- Eiter and Mannila, 1994: $O(n^2)$ time algorithm for the discrete Fréchet distance
- Buchin et al, 2007: $\Omega(n \log n)$ in the algebraic computation tree model
- Driemel et al, 2010: $O(cn/\varepsilon + cn \log n)$ time $(1+\varepsilon)$-approximation for $c-$packed curves

Faster Algorithms? 3-SUM hard?

Alt's Conjecture

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- Agarwal et al, 2013: $O(n^2 \frac{\log \log n}{\log n})$ time algorithm for the discrete Fréchet distance
- Buchin et al, 2014: $O(n^2 \sqrt{\log n} (\log \log n)^{3/2})$ expected time algorithm for the Fréchet distance

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- Bringmann, 2014: no subquadratic algorithm exists unless SETH fails

- Bringmann, Mulzer, 2015: no subquadratic algorithm for 1.399 approximation exists unless SETH fails

- Buchin et al. 2016: no subexponential algorithm for k curves exists unless SETH fails

Discrete Fréchet Distance

RUB

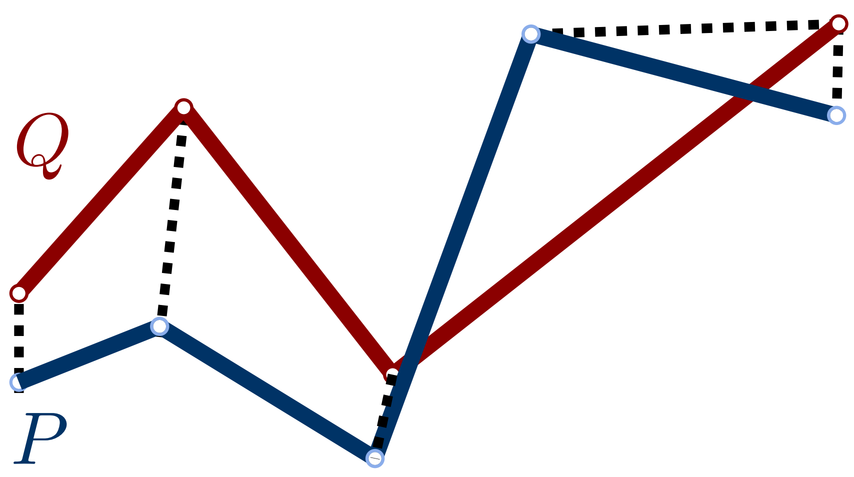
A **coupling** of $P = \langle p_1, \ldots p_n \rangle$ und $Q = \langle q_1, \ldots q_n \rangle$ is a sequence $C = \langle c_1, \ldots, c_m \rangle$ of pairs $c_l = (p_i, q_j)$ s.t.

- $c_1 = (p_1, q_1)$ and $c_m = (p_n, q_n)$
- $c_l = (p_i, q_j) \; \Rightarrow \; c_{l+1} \in \{(p_i, q_{j+1}), (p_{i+1}, q_j), (p_{i+1}, q_{j+1})\}$

Definition: $\delta_{dF}(P, Q) := \min_C \max_{(p_i, q_j) \in C} d(p_i, q_j)$

Approximation: $\delta_F(P, Q) \leq \delta_{dF}(P, Q) \leq \delta_F(P, Q) + d$

- closely related to **dynamic time warping**



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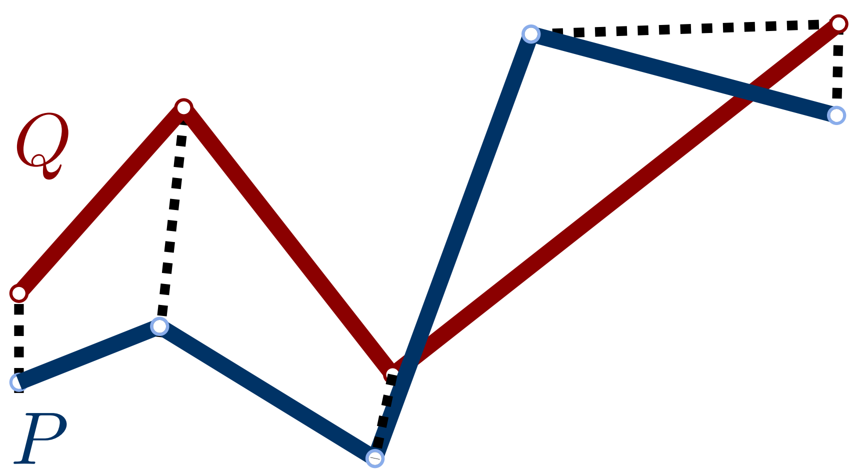
Definition: $\delta_{dF}(P, Q) := \min_C \max_{(p_i, q_j) \in C} d(p_i, q_j)$

Algorithm: compute cheapest path in distance matrix in $O(n^2)$ time [Eiter, Mannila, '94]



Q

209	128	57	81	38
117	72	43	117	135
76	63	113	153	183
41	52	140	156	202

P

Lower Bound by Bringmann

Conditional Lower Bound

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Strong Exponential Time Hypothesis (SETH): there is no $\epsilon > 0$ such that k-SAT has an $O(2^{(1-\epsilon)N})$ algorithm for all k.

Theorem: There is no $O(n^{2-\delta})$ algorithm for the Fréchet distance for any $\delta > 0$, unless SETH fails.

Proof: Reduction of $k-$CNF SAT to the Fréchet distance

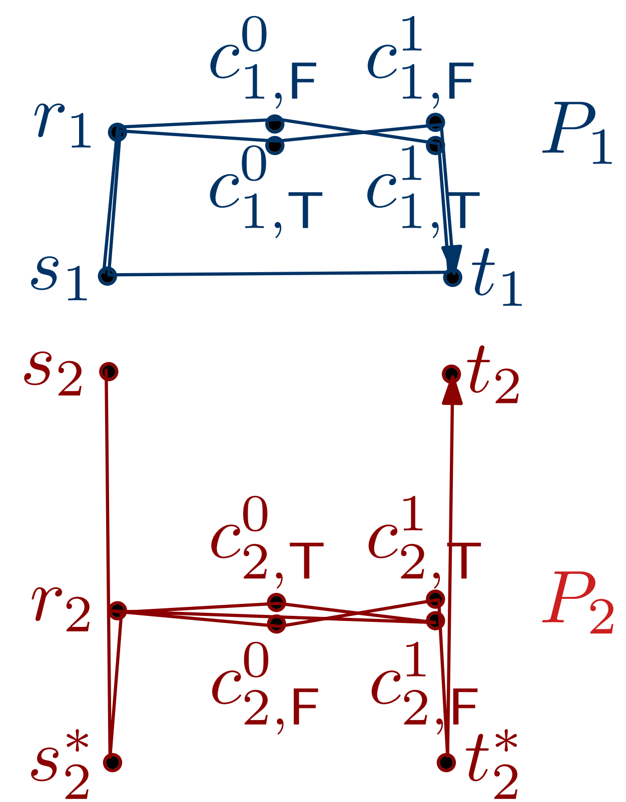
Reduction from k-CNF SAT

Given a CNF-SAT formula F with N variables and M clauses.

Construct two curves P_1, P_2 such that

$$d_F(P_1, P_2) \leq 1 \quad \Leftrightarrow \quad F \text{ is satisfiable.}$$

Split the vertices in halves V_1, V_2 and construct P_1, P_2 from all of their partial assignments A_1, A_2.



- for $i = 1, 2$ place points $r_i, c^{0|1}_{i,\mathsf{T}|\mathsf{F}}$
- for each assignment a_j of V_i and clause C_k choose the point $c^{k \bmod 2}_{i,\mathsf{sat}(a_j,C_k)}$
- let $AG(a_k) := r_k \circ \bigcirc_{i \in [M]} CG(a_k, i)$.

Lemma: $\delta_(AG(a_1), AG(a_2)) \leq 1$ iff (a_1, a_2) is a satisfying assignment.

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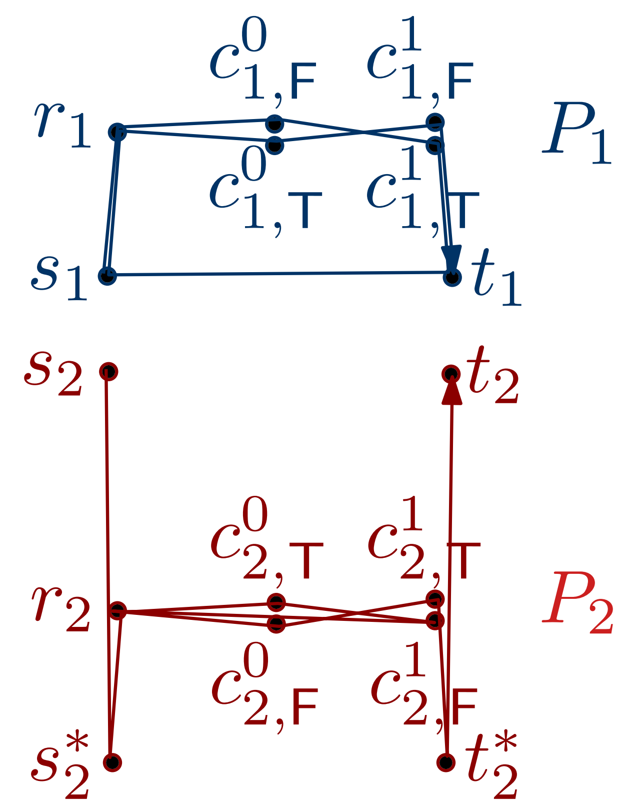

- for $i = 1, 2$ place points $r_i, c^{0|1}_{i,\mathsf{T}|\mathsf{F}}$
- for each assignment a_j of V_i and clause C_k choose the point $c^{k \bmod 2}_{i,\mathsf{sat}(a_j, C_k)}$
- for V_1 concatenate all assignments cyclic
- for V_2 concatenate all assignments sequential

$$P_1 := \bigcirc_{a_1 \in A_1} (s_1 \circ AG(a_1) \circ t_1),$$
$$P_2 := s_2 \circ s_2^* \circ \left(\bigcirc_{a_2 \in A_2} AG(a_2) \right) \circ t_2^* \circ t_2.$$

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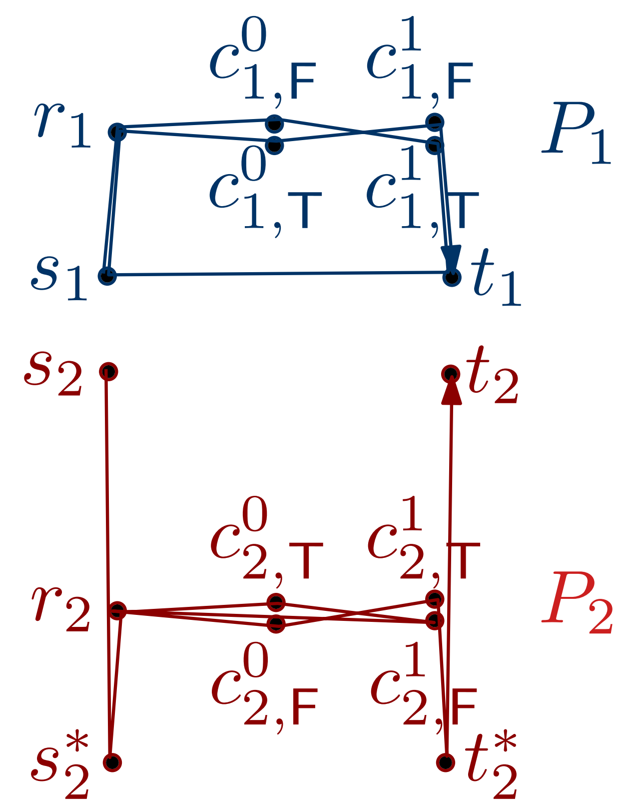

It is $|A_i| = O(2^{N/2})$, hence $n = |P_i| = O(M2^{N/2})$ and an $O(n^{2-\delta})$ algorithm would imply an $O(M^2 2^{(1-\delta/2)N})$ SAT algorithm