Lecture 2: Logic, Complexity Theory, and Their Application to Packing Problems

Helmut Alt, Freie Universität Berlin
at 10th Winter School on Computational Geometry
Amirkabir University of Technology, Tehran

1 March 2018

Mathematical Logic

- **idea:** making mathematical reasoning a subject of mathematics itself

Mathematical Logic

- **idea:** making mathematical reasoning a subject of mathematics itself
- **1st order predicate calculus for arithmetic**

Mathematical Logic

- **idea:** making mathematical reasoning a subject of mathematics itself
- **1st order predicate calculus for arithmetic**

  logical formulas consist of

  - *variables* a,b,c,....x,y,z,....

Mathematical Logic

- **idea:** making mathematical reasoning a subject of mathematics itself
- **1st order predicate calculus for arithmetic**

  logical formulas consist of

  - *variables* a,b,c,....x,y,z,....
  - *constant symbols* $0, 1$

Mathematical Logic

- **idea:** making mathematical reasoning a subject of mathematics itself
- **1st order predicate calculus for arithmetic**

  logical formulas consist of

  - *variables* a,b,c,....x,y,z,....
  - *constant symbols* $0, 1$
  - *function symbols* $+, *$

Mathematical Logic

- **idea:** making mathematical reasoning a subject of mathematics itself
- **1st order predicate calculus for arithmetic**

  logical formulas consist of

  - *variables* a,b,c,....x,y,z,....
  - *constant symbols* $0, 1$
  - *function symbols* $+, *$
  - *predicate symbols* $\leq, <, =, \neq$

Mathematical Logic

- **idea:** making mathematical reasoning a subject of mathematics itself
- **1st order predicate calculus for arithmetic**

  logical formulas consist of

  - *variables* a,b,c,....x,y,z,....
  - *constant symbols* $0, 1$
  - *function symbols* $+, *$
  - *predicate symbols* $\leq, <, =, \neq$
  - *quantifiers* $\forall, \exists$

Mathematical Logic

- **idea:** making mathematical reasoning a subject of mathematics itself
- **1st order predicate calculus for arithmetic**

  logical formulas consist of

  - *variables* a,b,c,....x,y,z,....
  - *constant symbols* $0, 1$
  - *function symbols* $+, *$
  - *predicate symbols* $\leq, <, =, \neq$
  - *quantifiers* $\forall, \exists$
  - *Boolean connectives* $\vee, \wedge, \neg$

Mathematical Logic

- **idea:** making mathematical reasoning a subject of mathematics itself
- **1st order predicate calculus for arithmetic**

  logical formulas consist of

  - *variables* a,b,c,....x,y,z,....
  - *constant symbols* $0, 1$
  - *function symbols* $+, *$
  - *predicate symbols* $\leq, <, =, \neq$
  - *quantifiers* $\forall, \exists$
  - *Boolean connectives* $\vee, \wedge, \neg$
  - *parentheses* $(,)$

Mathematical Logic

- **idea:** making mathematical reasoning a subject of mathematics itself
- **1st order predicate calculus for arithmetic**

  logical formulas consist of

  - *variables* a,b,c,....x,y,z,....
  - *constant symbols* $0, 1$
  - *function symbols* $+, *$
  - *predicate symbols* $\leq, <, =, \neq$
  - *quantifiers* $\forall, \exists$
  - *Boolean connectives* $\vee, \wedge, \neg$
  - *parentheses* $(,)$
- Examples:

Mathematical Logic

- **idea:** making mathematical reasoning a subject of mathematics itself
- **1st order predicate calculus for arithmetic**

  logical formulas consist of

  - *variables* a,b,c,....x,y,z,....
  - *constant symbols* $0, 1$
  - *function symbols* $+, *$
  - *predicate symbols* $\leq, <, =, \neq$
  - *quantifiers* $\forall, \exists$
  - *Boolean connectives* $\vee, \wedge, \neg$
  - *parentheses* $(,)$
- Examples:
  - $\forall x \exists y \; x + x = y$

Mathematical Logic

- **idea:** making mathematical reasoning a subject of mathematics itself
- **1st order predicate calculus for arithmetic**

  logical formulas consist of

  - *variables* a,b,c,....x,y,z,....
  - *constant symbols* $0, 1$
  - *function symbols* $+, *$
  - *predicate symbols* $\leq, <, =, \neq$
  - *quantifiers* $\forall, \exists$
  - *Boolean connectives* $\vee, \wedge, \neg$
  - *parentheses* $(,)$
- Examples:
  - $\forall x \exists y\ x + x = y$
  - $\exists x \exists y \exists z\ x \neq 0 \wedge y \neq 0 \wedge x * x * x + y * y * y = z * z * z$

Mathematical Logic

- **idea:** making mathematical reasoning a subject of mathematics itself
- **1st order predicate calculus for arithmetic**

  logical formulas consist of

  - *variables* a,b,c,....x,y,z,....
  - *constant symbols* $0, 1$
  - *function symbols* $+, *$
  - *predicate symbols* $\leq, <, =, \neq$
  - *quantifiers* $\forall, \exists$
  - *Boolean connectives* $\vee, \wedge, \neg$
  - *parentheses* $(,)$
- Examples:
  - $\forall x \exists y \; x + x = y$
  - $\exists x \exists y \exists z \; x \neq 0 \wedge y \neq 0 \wedge x * x * x + y * y * y = z * z * z$
  - $\forall q \exists p \forall x \forall y (p \geq q) \wedge (x \leq 1 \vee y \leq 1) \vee (x * y \neq p \wedge x * y \neq p + 2))$

Models

Models

A **Model** $\mathcal{M}$ assigns an underlying *universe* to variables and constant symbols, functions to function symbols, predicates to predicate symbols.

Models

A **Model** $\mathcal{M}$ assigns an underlying *universe* to variables and constant symbols, functions to function symbols, predicates to predicate symbols.

If in a formula all variables are quantified (a *sentence*), a model turns it into a proposition, which is **true** or **false**.

Models

A **Model** $\mathcal{M}$ assigns an underlying *universe* to variables and constant symbols, functions to function symbols, predicates to predicate symbols.

If in a formula all variables are quantified (a *sentence*), a model turns it into a proposition, which is **true** or **false**.

Examples:
for model ($\mathbb{N}$, addition, multiplication)

Models

A **Model** $\mathcal{M}$ assigns an underlying *universe* to variables and constant symbols, functions to function symbols, predicates to predicate symbols.

If in a formula all variables are quantified (a *sentence*), a model turns it into a proposition, which is **true** or **false**.

Examples:
for model ($\mathbb{N}$, addition, multiplication)

- $\forall x \exists y \; x + x = y$

Models

A **Model** $\mathcal{M}$ assigns an underlying *universe* to variables and constant symbols, functions to function symbols, predicates to predicate symbols.

If in a formula all variables are quantified (a *sentence*), a model turns it into a proposition, which is **true** or **false**.

Examples:
for model ($\mathbb{N}$, addition, multiplication)

- $\forall x \exists y \; x + x = y$ **true**

Models

A **Model** $\mathcal{M}$ assigns an underlying *universe* to variables and constant symbols, functions to function symbols, predicates to predicate symbols.

If in a formula all variables are quantified (a *sentence*), a model turns it into a proposition, which is **true** or **false**.

Examples:
for model ($\mathbb{N}$, addition, multiplication)

- $\forall x \exists y \; x + x = y$ **true**
- $\exists x \exists y \exists z \; x \neq 0 \wedge y \neq 0 \wedge x^3 + y^3 = z^3$

Models

A **Model** $\mathcal{M}$ assigns an underlying *universe* to variables and constant symbols, functions to function symbols, predicates to predicate symbols.

If in a formula all variables are quantified (a *sentence*), a model turns it into a proposition, which is **true** or **false**.

Examples:
for model ($\mathbb{N}$, addition, multiplication)

- $\forall x \exists y \; x + x = y$ **true**
- $\exists x \exists y \exists z \; x \neq 0 \wedge y \neq 0 \wedge x^3 + y^3 = z^3$ **false**

Models

A **Model** $\mathcal{M}$ assigns an underlying *universe* to variables and constant symbols, functions to function symbols, predicates to predicate symbols.

If in a formula all variables are quantified (a *sentence*), a model turns it into a proposition, which is **true** or **false**.

Examples:
for model ($\mathbb{N}$, addition, multiplication)

- $\forall x \exists y\ x + x = y$ **true**
- $\exists x \exists y \exists z\ x \neq 0 \wedge y \neq 0 \wedge x^3 + y^3 = z^3$ **false**
- $\forall q \exists p \forall x \forall y (p \geq q) \wedge ((x \leq 1 \vee y \leq 1) \vee (x * y \neq p \wedge x * y \neq p+2))$

Models

A **Model** $\mathcal{M}$ assigns an underlying *universe* to variables and constant symbols, functions to function symbols, predicates to predicate symbols.

If in a formula all variables are quantified (a *sentence*), a model turns it into a proposition, which is **true** or **false**.

Examples:
for model ($\mathbb{N}$, addition, multiplication)

- $\forall x \exists y \; x + x = y$ **true**
- $\exists x \exists y \exists z \; x \neq 0 \wedge y \neq 0 \wedge x^3 + y^3 = z^3$ **false**
- $\forall q \exists p \forall x \forall y (p \geq q) \wedge ((x \leq 1 \vee y \leq 1) \vee (x * y \neq p \wedge x * y \neq p + 2))$ **unknown**

Models

A **Model** $\mathcal{M}$ assigns an underlying *universe* to variables and constant symbols, functions to function symbols, predicates to predicate symbols.

If in a formula all variables are quantified (a *sentence*), a model turns it into a proposition, which is **true** or **false**.

Examples:
for model ($\mathbb{N}$, addition, multiplication)

- $\forall x \exists y \; x + x = y$ **true**
- $\exists x \exists y \exists z \; x \neq 0 \wedge y \neq 0 \wedge x^3 + y^3 = z^3$ **false**
- $\forall q \exists p \forall x \forall y (p \geq q) \wedge ((x \leq 1 \vee y \leq 1) \vee (x * y \neq p \wedge x * y \neq p + 2))$ **unknown**

Set of all true sentences: $\textbf{Th}(\mathcal{M})$.

Models

A **Model** $\mathcal{M}$ assigns an underlying *universe* to variables and constant symbols, functions to function symbols, predicates to predicate symbols.

If in a formula all variables are quantified (a *sentence*), a model turns it into a proposition, which is **true** or **false**.

Examples:
for model ($\mathbb{N}$, addition, multiplication)

- $\forall x \exists y \; x + x = y$ **true**
- $\exists x \exists y \exists z \; x \neq 0 \wedge y \neq 0 \wedge x^3 + y^3 = z^3$ **false**
- $\forall q \exists p \forall x \forall y (p \geq q) \wedge ((x \leq 1 \vee y \leq 1) \vee (x * y \neq p \wedge x * y \neq p + 2))$ **unknown**

Set of all true sentences: $\mathbf{Th}(\mathcal{M})$.

e.g.,

$\mathbf{Th}(\mathbb{N}, +, *)$ Theory of **arithmetic**

Is Mathematical Truth Decidable?

Is Mathematical Truth Decidable?

- Is $\mathbf{Th}(\mathbb{N}, +, *)$ *decidable*,

Is Mathematical Truth Decidable?

- Is $\mathbf{Th}(\mathbb{N}, +, *)$ *decidable*, i.e., is there an algorithm which for a given sentence can determine whether it is true or false?

Is Mathematical Truth Decidable?

- Is **Th**$(\mathbb{N}, +, *)$ *decidable*, i.e., is there an algorithm which for a given sentence can determine whether it is true or false?
- Early 20th century mathematicians around **Hilbert** believed that **Th**$(\mathbb{N}, +, *)$ and other mathematical theories can be completely represented by a formal system of *axioms* and *rules of inference*, i.e., exactly the true propositions can be *deduced*.

Is Mathematical Truth Decidable?

- Is $\mathbf{Th}(\mathbb{N}, +, *)$ *decidable*, i.e., is there an algorithm which for a given sentence can determine whether it is true or false?
- Early 20th century mathematicians around **Hilbert** believed that $\mathbf{Th}(\mathbb{N}, +, *)$ and other mathematical theories can be completely represented by a formal system of *axioms* and *rules of inference*, i.e., exactly the true propositions can be *deduced*.
- Disproved by **Gödel's Incompleteness Theorem, 1931** stating that there cannot be any complete and consistent system for the theory of arithmetic.

Decidable Theories

Decidable Theories

- **Th**$(\mathbb{N}, +)$

Decidable Theories

- **Th**$(\mathbb{N}, +)$ is **decidable!**

Decidable Theories

- **Th**$(\mathbb{N}, +)$ is **decidable!**
  Presburger 1929

Decidable Theories

- **Th**$(\mathbb{N}, +)$ is **decidable!**
  Presburger 1929 "Presburger-arithmetic"

Decidable Theories

- $\mathbf{Th}(\mathbb{N}, +)$ is **decidable!**
  Presburger 1929 "Presburger-arithmetic"

- $\mathbf{Th}(\mathbb{R}, +, *)$

Decidable Theories

- $\mathbf{Th}(\mathbb{N}, +)$ is **decidable!**
  Presburger 1929 "Presburger-arithmetic"
- $\mathbf{Th}(\mathbb{R}, +, *)$
  - $\forall x \exists y \; x + x = y$
  - $\exists x \exists y \exists z \; x \neq 0 \wedge y \neq 0 \wedge x^3 + y^3 = z^3$
  - $\forall q \exists p \forall x \forall y (p \geq q) \wedge ((x \leq 1 \vee y \leq 1) \vee (x * y \neq p \wedge x * y \neq p + 2))$

  have truth values *true*, *true*, *false* obviously.

Decidable Theories

- $\mathbf{Th}(\mathbb{N}, +)$ is **decidable!**
  Presburger 1929 "Presburger-arithmetic"
- $\mathbf{Th}(\mathbb{R}, +, *)$
  - $\forall x \exists y \; x + x = y$
  - $\exists x \exists y \exists z \; x \neq 0 \wedge y \neq 0 \wedge x^3 + y^3 = z^3$
  - $\forall q \exists p \forall x \forall y (p \geq q) \wedge ((x \leq 1 \vee y \leq 1) \vee (x * y \neq p \wedge x * y \neq p + 2))$

  have truth values *true*, *true*, *false* obviously.

Decidable Theories

- **Th**$(\mathbb{N}, +)$ is **decidable!**
  Presburger 1929 "Presburger-arithmetic"
- **Th**$(\mathbb{R}, +, *)$ is **decidable!**
  Tarski 1948

Decidable Theories

- $\mathbf{Th}(\mathbb{N}, +)$ is **decidable!**
  Presburger 1929 "Presburger-arithmetic"
- $\mathbf{Th}(\mathbb{R}, +, *)$ is **decidable!**
  Tarski 1948 "Tarski formulas"

Complexity

- $\mathbf{Th}(\mathbb{N}, +)$

Complexity

- **Th**$(\mathbb{N}, +)$
  Lower bound: Fischer, Rabin 1974

Complexity

- **Th**$(\mathbb{N}, +)$
  Lower bound: Fischer, Rabin 1974
  For the decision of a formula of length n any algorithm needs at least time doubly exponential in n.

Complexity

- **Th**$(\mathbb{N}, +)$
  Lower bound: Fischer, Rabin 1974
  For the decision of a formula of length n any algorithm needs at least time doubly exponential in n.
- **Th**$(\mathbb{R}, +, *)$

Complexity

- **Th**$(\mathbb{N}, +)$
  Lower bound: Fischer, Rabin 1974
  For the decision of a formula of length n any algorithm needs at least time doubly exponential in n.
- **Th**$(\mathbb{R}, +, *)$
  can be decided in (singly) exponential time.

Complexity

- **Th**$(\mathbb{N}, +)$
  Lower bound: Fischer, Rabin 1974
  For the decision of a formula of length n any algorithm needs at least time doubly exponential in n.
- **Th**$(\mathbb{R}, +, *)$
  can be decided in (singly) exponential time.
  Grigoriev, Vorobjov 1988 Renegar 1992.

Complexity

- **Th**$(\mathbb{N}, +)$
  Lower bound: Fischer, Rabin 1974
  For the decision of a formula of length n any algorithm needs at least time doubly exponential in n.
- **Th**$(\mathbb{R}, +, *)$
  can be decided in (singly) exponential time.
  Grigoriev, Vorobjov 1988 Renegar 1992.
- **the existential theory of the reals** $\exists$**Th**$(\mathbb{R}, +, *)$

Complexity

- **Th**$(\mathbb{N}, +)$
  Lower bound: Fischer, Rabin 1974
  For the decision of a formula of length n any algorithm needs at least time doubly exponential in n.
- **Th**$(\mathbb{R}, +, *)$
  can be decided in (singly) exponential time.
  Grigoriev, Vorobjov 1988 Renegar 1992.
- **the existential theory of the reals** $\exists$**Th**$(\mathbb{R}, +, *)$
  i.e., sentences of the form $\exists x_1 \cdots \exists x_k \phi(x_1, ..., x_k)$

Complexity

- **Th**$(\mathbb{N}, +)$
  Lower bound: Fischer, Rabin 1974
  For the decision of a formula of length n any algorithm needs at least time doubly exponential in n.
- **Th**$(\mathbb{R}, +, *)$
  can be decided in (singly) exponential time.
  Grigoriev, Vorobjov 1988 Renegar 1992.
- **the existential theory of the reals** $\exists$**Th**$(\mathbb{R}, +, *)$
  i.e., sentences of the form $\exists x_1 \cdots \exists x_k \phi(x_1, ..., x_k)$
  can be decided in PSPACE.

Complexity

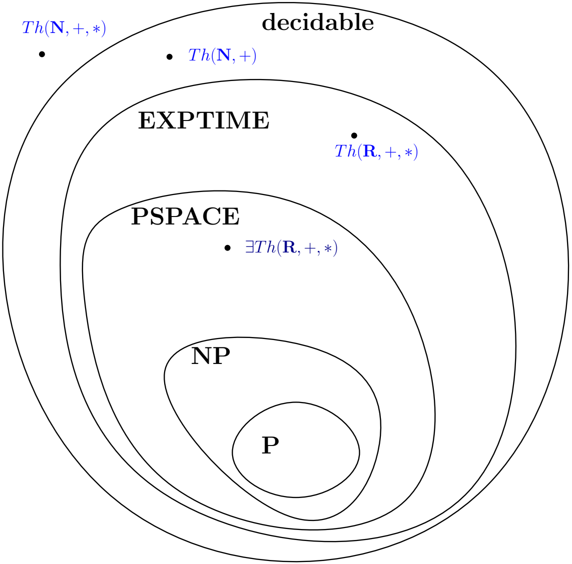
- **Th**$(\mathbb{N}, +)$
  Lower bound: Fischer, Rabin 1974
  For the decision of a formula of length n any algorithm needs at least time doubly exponential in n.
- **Th**$(\mathbb{R}, +, *)$
  can be decided in (singly) exponential time.
  Grigoriev, Vorobjov 1988 Renegar 1992.
- **the existential theory of the reals** $\exists$**Th**$(\mathbb{R}, +, *)$
  i.e., sentences of the form $\exists x_1 \cdots \exists x_k \phi(x_1, ..., x_k)$
  can be decided in PSPACE. Canny 1988

Complexity

- **Th**$(\mathbb{N}, +)$
  Lower bound: Fischer, Rabin 1974
  For the decision of a formula of length n any algorithm needs at least time doubly exponential in n.
- **Th**$(\mathbb{R}, +, *)$
  can be decided in (singly) exponential time.
  Grigoriev, Vorobjov 1988 Renegar 1992.
- **the existential theory of the reals** $\exists$**Th**$(\mathbb{R}, +, *)$
  i.e., sentences of the form $\exists x_1 \cdots \exists x_k \phi(x_1, ..., x_k)$
  can be decided in PSPACE. Canny 1988

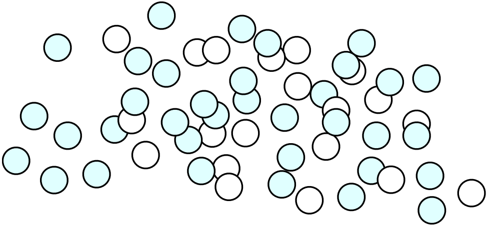
Complexity Classes

Complexity Classes

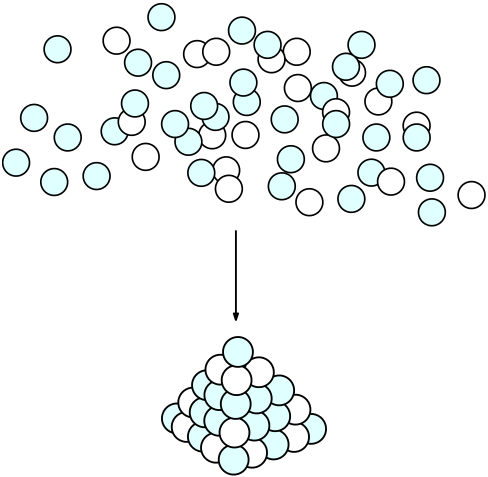


Packing

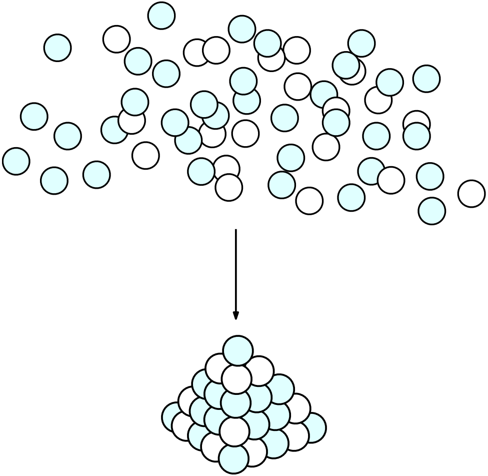
Packing



Packing



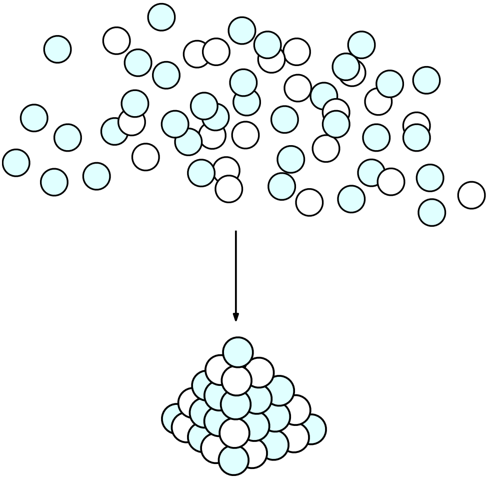
Packing



Kepler's conjecture

Kepler 1611, Harriot 1591, Raleigh

Packing



Kepler's conjecture — Kepler 1611, Harriot 1591, Raleigh

Proof by — Hales 2014

Surprising Open Problems

Surprising Open Problems

The sausage conjecture Fejes Tóth 1975

Surprising Open Problems

The sausage conjecture Fejes Tóth 1975

packing of n unit spheres in dimension d

Surprising Open Problems

The sausage conjecture Fejes Tóth 1975

packing of n unit spheres in dimension d

Is the sausage packing optimal, i.e., minimizing the convex hull?

Surprising Open Problems

The sausage conjecture Fejes Tóth 1975

packing of n unit spheres in dimension d

Is the sausage packing optimal, i.e., minimizing the convex hull?

Obviously not true for $d = 2$, even for $n = 3$:

Surprising Open Problems

The sausage conjecture Fejes Tóth 1975

packing of n unit spheres in dimension d

Is the sausage packing optimal, i.e., minimizing the convex hull?

Obviously not true for $d = 2$, even for $n = 3$: sausage catastrophe !

Surprising Open Problems

The sausage conjecture Fejes Tóth 1975

packing of n unit spheres in dimension d

Is the sausage packing optimal, i.e., minimizing the convex hull?

Obviously not true for $d = 2$, even for $n = 3$: sausage catastrophe !

$d = 3$: true for $n \leq 4$, false for $n \geq 56$

Surprising Open Problems

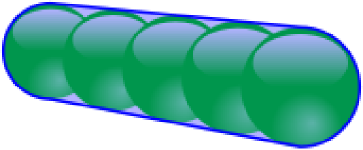
The sausage conjecture Fejes Tóth 1975

packing of n unit spheres in dimension d

Is the sausage packing optimal, i.e., minimizing the convex hull?

Obviously not true for $d = 2$, even for $n = 3$: sausage catastrophe !

$d = 3$: true for $n \leq 4$, false for $n \geq 56$



Surprising Open Problems

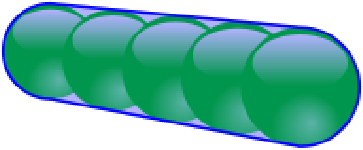
The sausage conjecture Fejes Tóth 1975

packing of n unit spheres in dimension d

Is the sausage packing optimal, i.e., minimizing the convex hull?

Obviously not true for $d = 2$, even for $n = 3$: sausage catastrophe !

$d = 3$: true for $n \leq 4$, false for $n \geq 56$



sausage conjecture is true for $d \geq 42$ Betke, Henk, Wills 1998

Surprising Open Problems

Surprising Open Problems

Smallest circle containing n unit circles

Surprising Open Problems

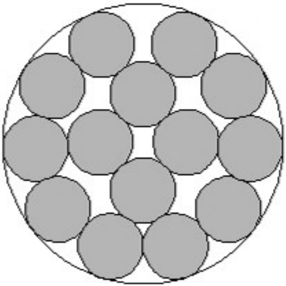
Smallest circle containing n unit circles

Is the following container smallest for $n = 14$?

Surprising Open Problems

Smallest circle containing n unit circles

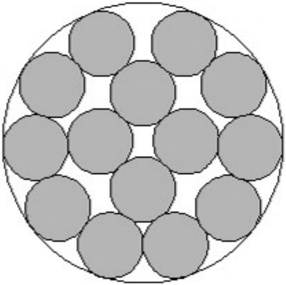
Is the following container smallest for $n = 14$?



Surprising Open Problems

Smallest circle containing n unit circles

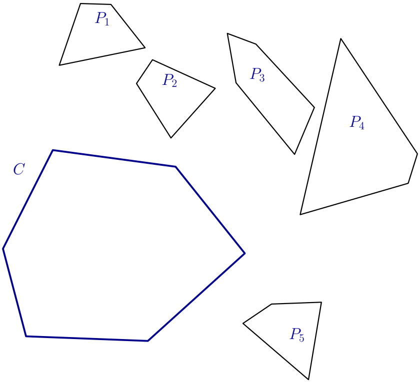
Is the following container smallest for $n = 14$?



http://www2.stetson.edu/~efriedma/packing.html
http://www.packomania.com/

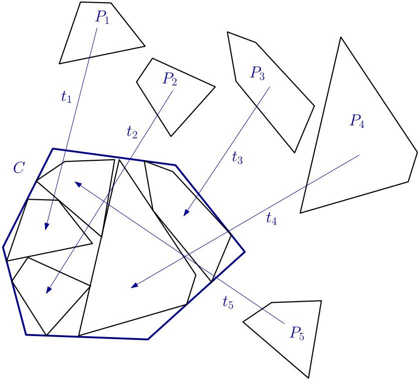
Application of $Th(\mathbb{R}, +, *)$ to Packing Problems

given: k polygons $P_1, \dots, P_k$, polygonal container C
question: can the polygons be put without overlap into C by translations?



Application of $Th(\mathbb{R}, +, *)$ to Packing Problems

given: k polygons $P_1, ..., P_k$, polygonal container C
question: can the polygons be put without overlap into C by translations?



Application of $Th(\mathbb{R}, +, *)$ to Packing Problems

Application of $Th(\mathbb{R}, +, *)$ to Packing Problems

Variables:

Coordinates of translation vectors $t_1, ..., t_k$, existentially quantified.

Application of $Th(\mathbb{R}, +, *)$ to Packing Problems

Variables:
Coordinates of translation vectors $t_1, ..., t_k$, existentially quantified.

Conditions

Application of $Th(\mathbb{R}, +, *)$ to Packing Problems

Variables:
Coordinates of translation vectors $t_1, ..., t_k$, existentially quantified.

Conditions

1. Any two (translated) objects may not intersect.

Application of $Th(\mathbb{R}, +, *)$ to Packing Problems

Variables:

Coordinates of translation vectors $t_1, ..., t_k$, existentially quantified.

Conditions

1. Any two (translated) objects may not intersect.
2. All objects must be inside the container.

Details of the conditions

- two polygons P,Q don't intersect $\Leftrightarrow$

Details of the conditions

- two polygons P,Q don't intersect $\Leftrightarrow$

  no vertex of Q is inside P $\wedge$

Details of the conditions

- two polygons P,Q don't intersect $\Leftrightarrow$

  no vertex of Q is inside P $\wedge$
  no vertex of P is inside Q $\wedge$

Details of the conditions

- two polygons P,Q don't intersect $\Leftrightarrow$

  no vertex of Q is inside P $\wedge$
  no vertex of P is inside Q $\wedge$
  no two edges intersect

Details of the conditions

- two polygons P,Q don't intersect $\Leftrightarrow$

  no vertex of Q is inside P $\wedge$
  no vertex of P is inside Q $\wedge$
  no two edges intersect

- point q inside convex polygon P $\Leftrightarrow$

Details of the conditions

- two polygons P,Q don't intersect $\Leftrightarrow$

  no vertex of Q is inside P $\wedge$
  no vertex of P is inside Q $\wedge$
  no two edges intersect

- point q inside convex polygon P $\Leftrightarrow$

  q right of all edges $\overrightarrow{pp'}$ if the polygon is directed clockwise.

Details of the conditions

- two polygons P,Q don't intersect $\Leftrightarrow$

  no vertex of Q is inside P $\wedge$
  no vertex of P is inside Q $\wedge$
  no two edges intersect

- point q inside convex polygon P $\Leftrightarrow$

  q right of all edges $\overrightarrow{pp'}$ if the polygon is directed clockwise.

- q right of $\overrightarrow{pp'}$ $\Leftrightarrow$

Details of the conditions

- two polygons P,Q don't intersect $\Leftrightarrow$

  no vertex of Q is inside P $\wedge$
  no vertex of P is inside Q $\wedge$
  no two edges intersect

- point q inside convex polygon P $\Leftrightarrow$

  q right of all edges $\overrightarrow{pp'}$ if the polygon is directed clockwise.

- q right of $\overrightarrow{pp'}$ $\Leftrightarrow$

  $(p_x - p'_x)q_y \leq (p_y - p'_y)q_x + p_x p'_y - p_y p'_x$

Packing under Rigid Motions

i.e., besides translations also rotations of the objects are allowed:

in addition to translation, vertices of object P_i are multiplied with some matrix $\begin{pmatrix} \cos\phi_i & \sin\phi_i \\ -\sin\phi_i & \cos\phi_i \end{pmatrix}$

realized by additional variables
$c_i, s_i \qquad i = 1, 2, \ldots$

and additional conditions
$c_i^2 + s_i^2 = 1 \qquad i = 1, 2, \ldots.$

Minimum area container

Minimum area container

The optimization problem of, given a set of objects, finding the minimum area container (from a given algebraically described set of possible containers) seems to require a universal quantifier:

Minimum area container

The optimization problem of, given a set of objects, finding the minimum area container (from a given algebraically described set of possible containers) seems to require a universal quantifier:

3. All admissable containers have an area at least as large as the specified one.

But can be formulated in $Th(\mathbb{R}, +, *)$, so it is in EXPTIME.

Complexity of Packing Problems

Complexity of Packing Problems

even very simple ones are NP-hard!

Complexity of Packing Problems

even very simple ones are NP-hard!

all objects are simple polygons, containers are convex polygons

O: optimization problem, D: decision problem for given container

T: under translations, R: under rigid motions

Complexity of Packing Problems

- even very simple ones are NP-hard!
- all objects are simple polygons, containers are convex polygons
- O: optimization problem, D: decision problem for given container
- T: under translations, R: under rigid motions