Lecture 1: Definition and Complexity of Packing Problems

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at 10th Winter School on Computational Geometry
Amirkabir University of Technology, Tehran

28 February 2018

Packing Problems

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given:
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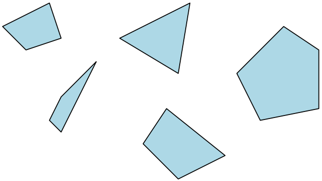

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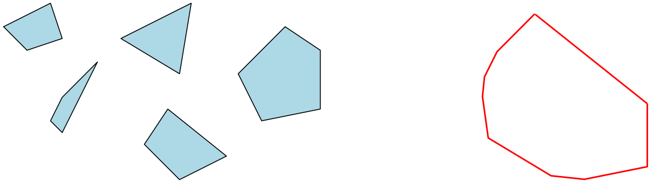

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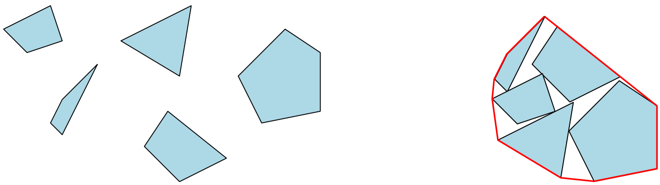

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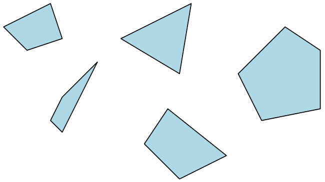

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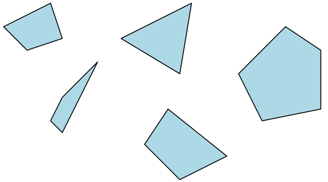

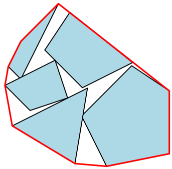

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- **objective function:**
  area/volume, circumference/surface area, diameter

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It is well known: NP $\subseteq$ PSPACE, the class of all problems solvable in polynomial *space*.

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cooperation with apparel industry

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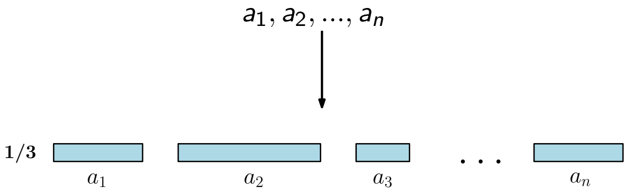

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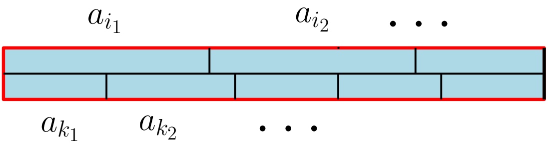
partition $a_{i_1} + \ldots + a_{i_l} = a_{k_1} + \ldots + a_{k_m}$ is possible, iff



perfect packing is possible

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- also, it follows or can easily be shown:
  - packing of rectangles/arbitrary polygons into rectangular/polygonal containers under translation/rigid motion is NP-hard.
  - for rectangular/convex polygonal containers the corresponding optimization problems are NP-hard.

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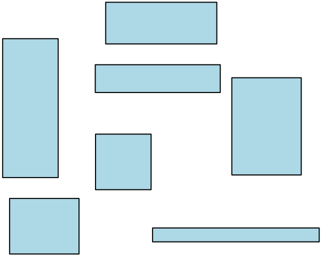

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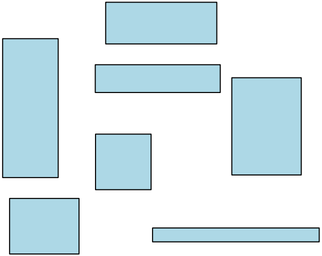

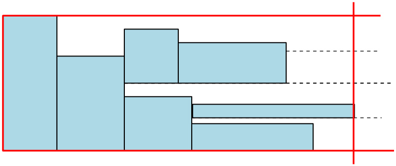

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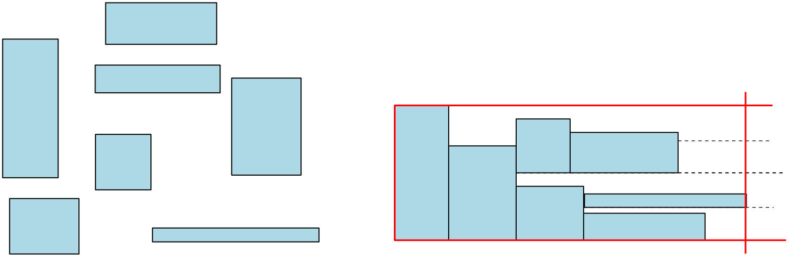

also possible:
3-approximation for packing axes-parallel rectangles under rigid motion

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under rigid motion, where the container is an axes-parallel rectangle.

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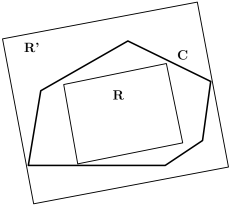

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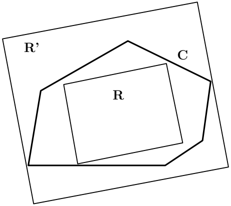

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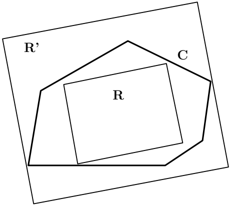

It follows: Any α-approximative packing algorithm for rectangular objects can be used to get a 4α-approximation algorithm for convex polygonal objects.