RUB

Algorithms for Comparing Curves and Surfaces

2: Other ways of searching the free space

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WSCG 2017

Free Space Diagram

- in 1992 Alt and Godau proposed the *free space diagram* as a means to compute the *Fréchet distance*
- basis for all subsequent algorithms

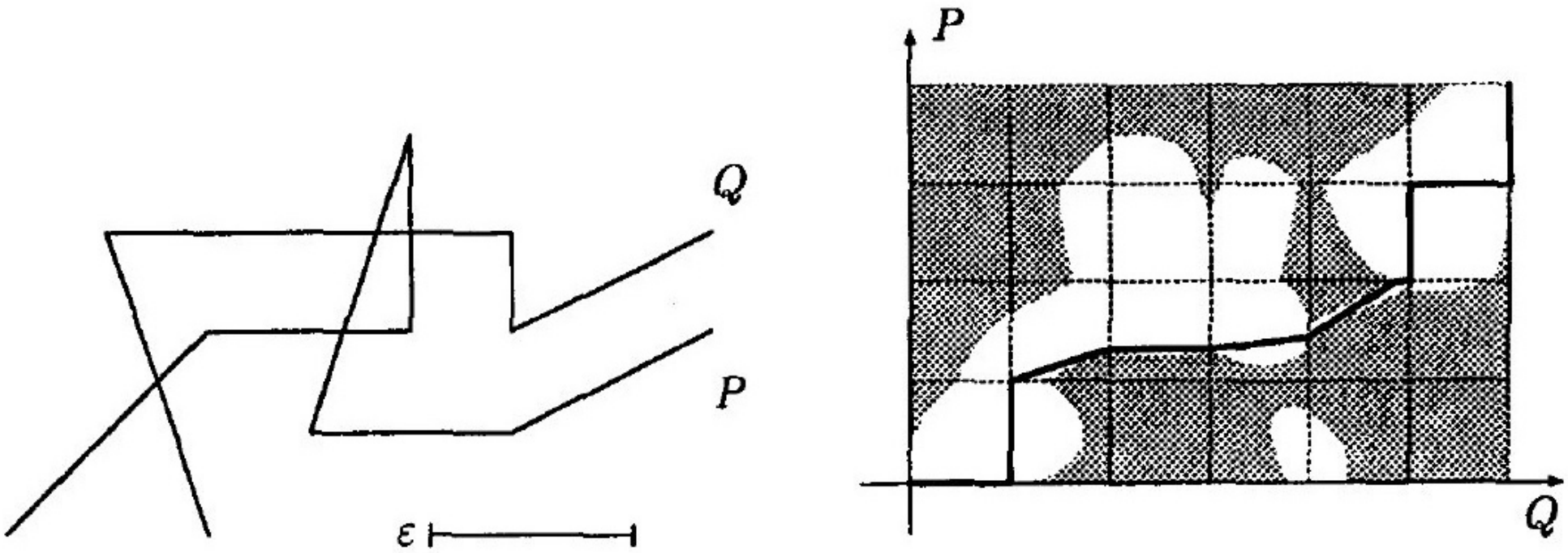

Fig. 3. Diagram for polygonal chains P, Q and the given ε

Algorithm of Alt & Godau

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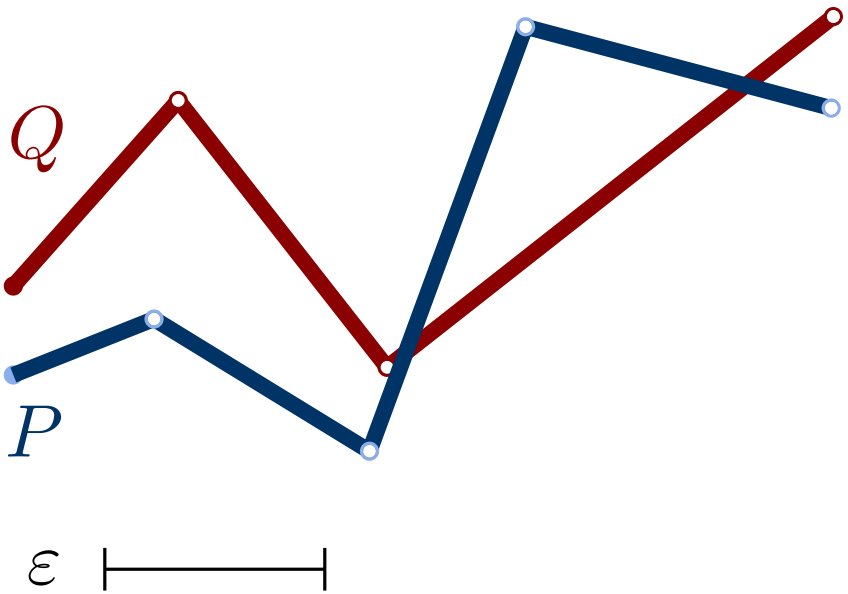
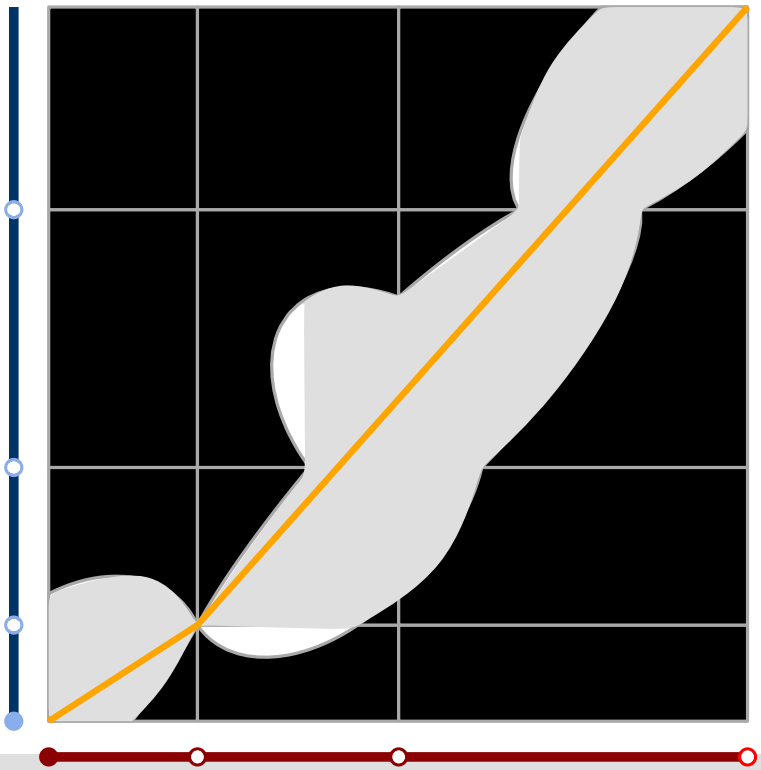
Let P, Q be two polygonal curves of size n.

- **decision problem:** Is $d_F(P, Q) \leq \varepsilon$?
  Solve by searching for a monotone path in the free space in $O(n^2)$ time
- **computation problem:** What is $d_F(P, Q)$?
  Solve by searching over a set of critical values in $O(n^2 \log n)$ time





Searching the Free Space Faster or Better

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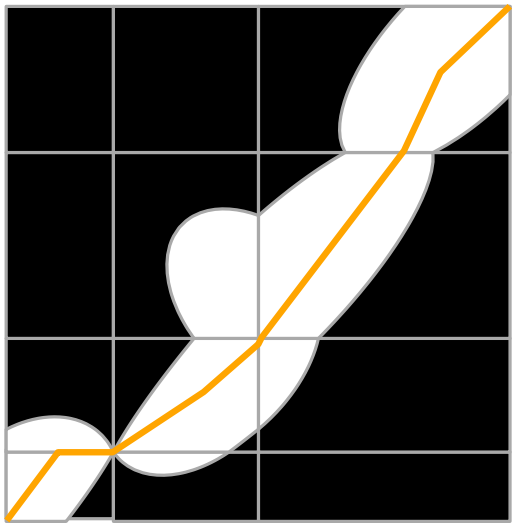
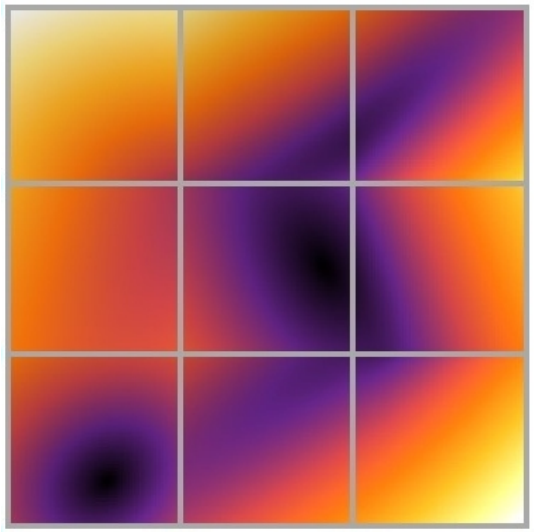
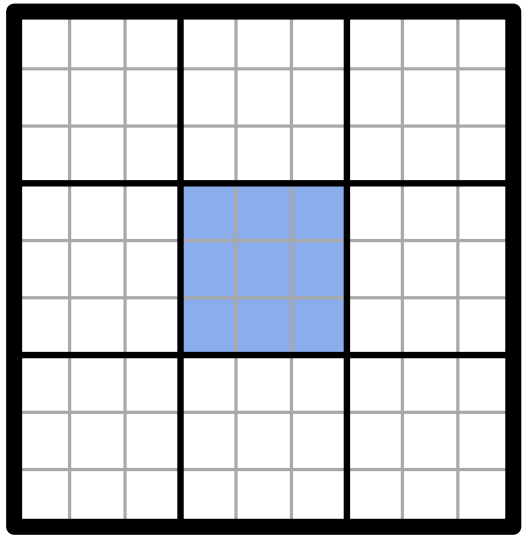
- locally correct [2012, Buchin et al.]
- lexicographic [2014, Rote]
- retractable [2013, Buchin et al.]
- faster [2014, Buchin et al.]







Searching for an Optimal Matching

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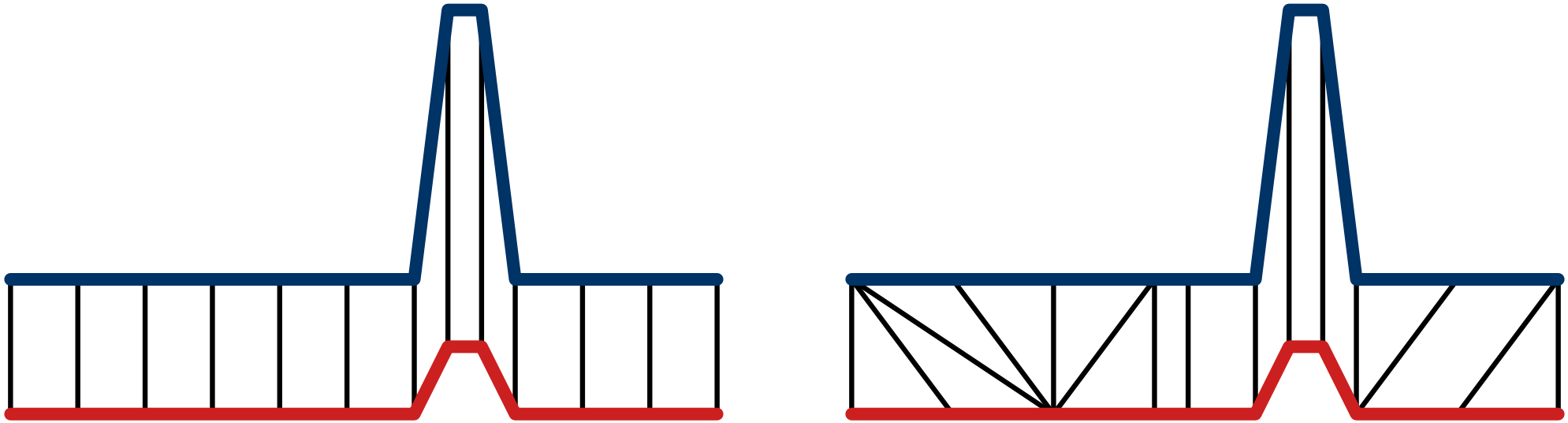
Weakness: many matchings may realize the Fréchet distance



Goal: find optimal matchings

now: lexicographic Fréchet matchings

Lexicographic Fréchet Matchings

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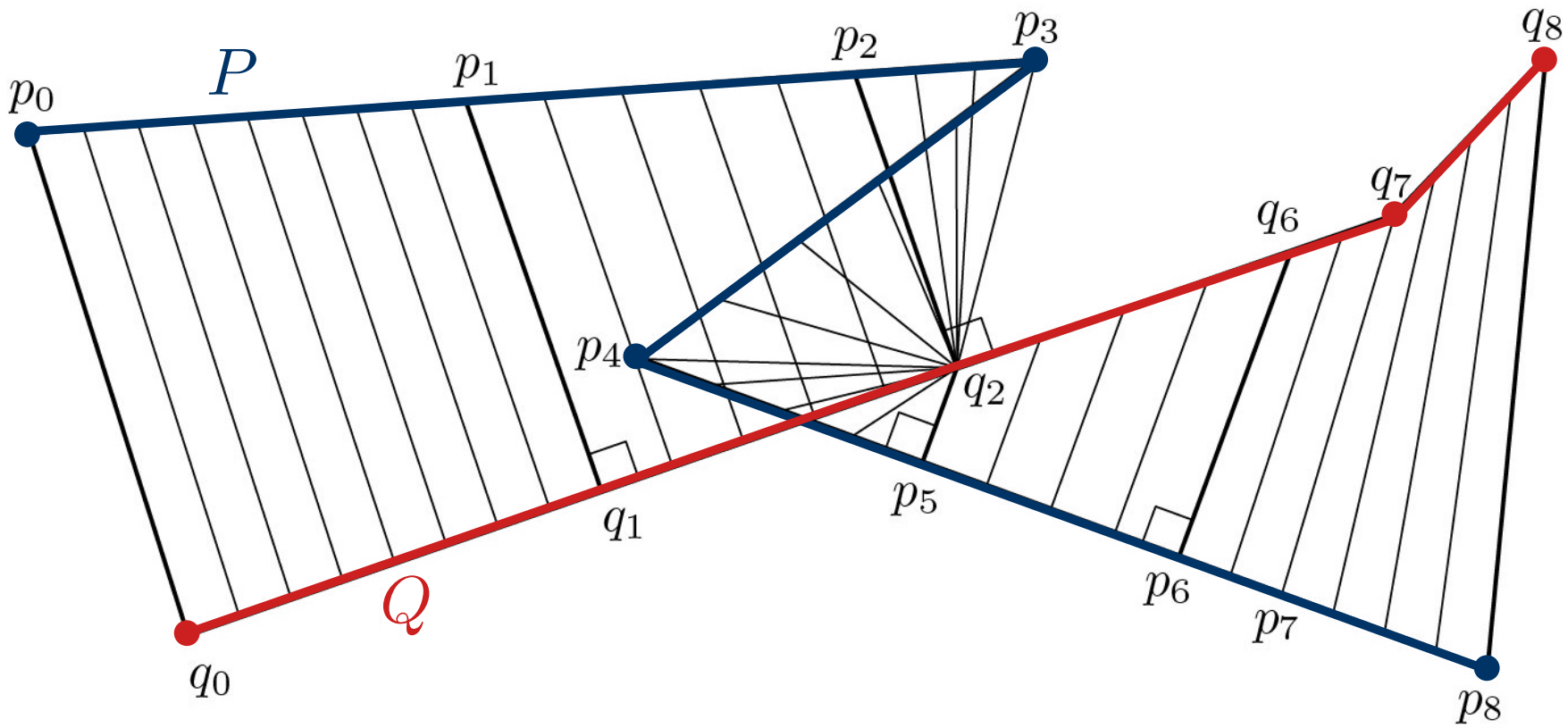
- $P\colon [0, L_P] \to \mathbb{R}^d$ and $Q\colon [0, L_Q] \to \mathbb{R}^d$ polygonal curves
- reparametrize by $\alpha\colon [0, M] \to [0, L_P]$ and $\beta\colon [0, M] \to [0, L_Q]$
- simultaneous traversal $(P(\alpha(t)), Q(\beta(t)))$ for $0 \leq t \leq M$
- Fréchet distance minimizes $\max_t ||P(\alpha(t)) - Q(\beta(t))||$



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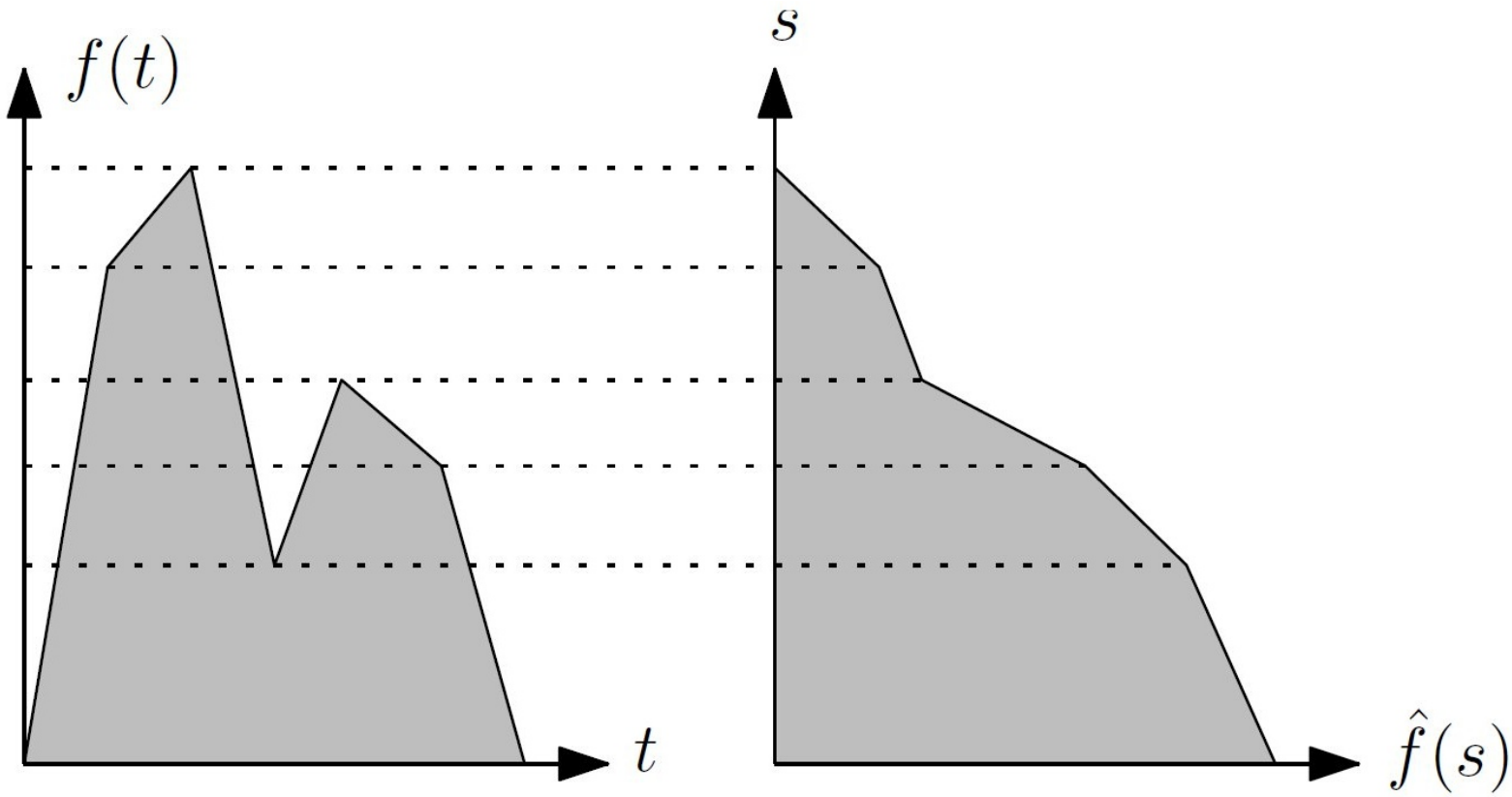
- reparametrize by $\alpha\colon [0, M] \to [0, L_P]$ and $\beta\colon [0, M] \to [0, L_Q]$
- now consider $f\colon [0, M] \to \mathbb{R}_{\geq 0}, \quad t \mapsto ||P(\alpha(t)) - Q(\beta(t))||$
- and its profile $\hat{f}\colon \mathbb{R} \to \mathbb{R}_{\geq 0}, \quad s \mapsto \mu(\{t | f(t) \geq s\})$



Lexicographic Fréchet Matchings

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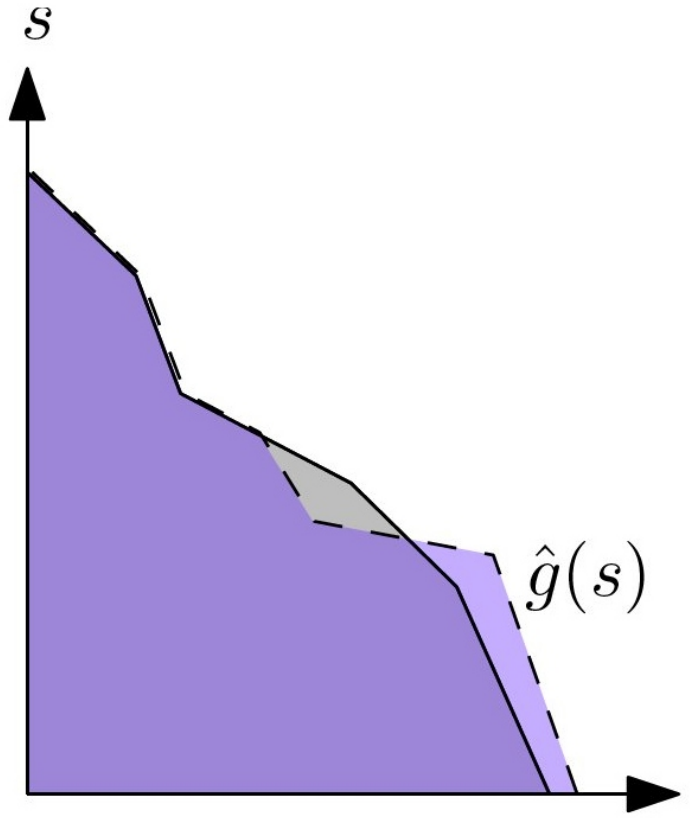
- compare lexicographically

  $f \leq_{lex} g \Leftrightarrow$

  $\exists s_0 [\forall s \geq s_0 : \hat{f}(s) = \hat{g}(s) \ \wedge$

  $\forall \varepsilon > 0 \exists s \in (s_0 - \varepsilon, s_0) \hat{f}(s) < \hat{g}(s)]$
- requires normalization:
  speed of reparameterizations α, β
  is bounded by 1



Lexicographic Fréchet Matchings

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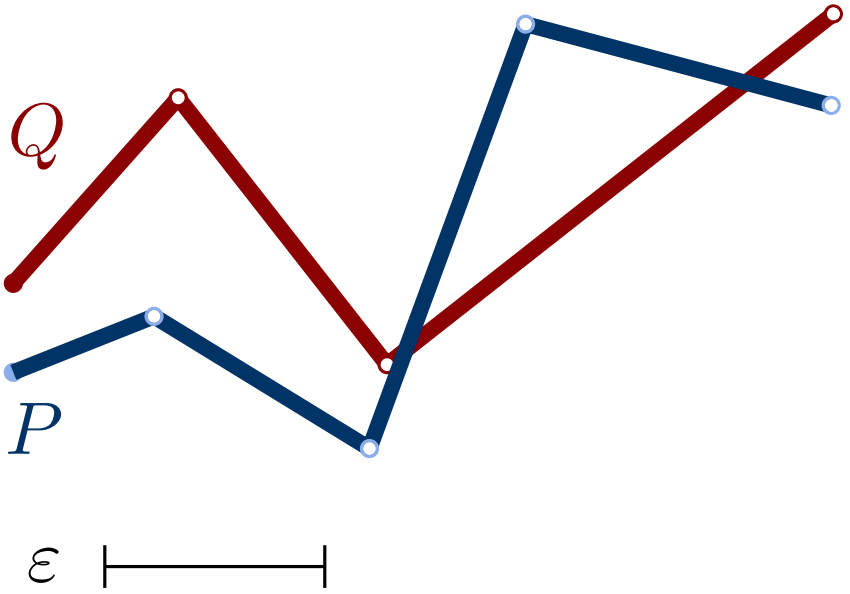
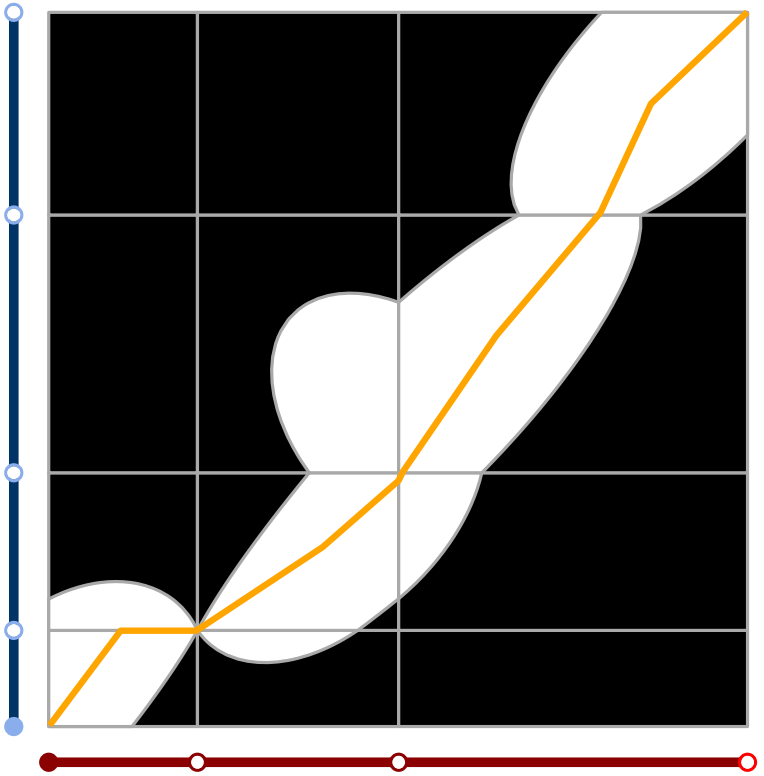
in the Free Space Diagram

- a monotone path $c : [0, M] \to F_\varepsilon$ with lowest profile
- speed constraint: c is Lipschitz continuous w.r.t L_∞-norm





general idea to find such a curve:
follow steepest descent path if not constrained by critical values

Following Steepest Descent within a Cell

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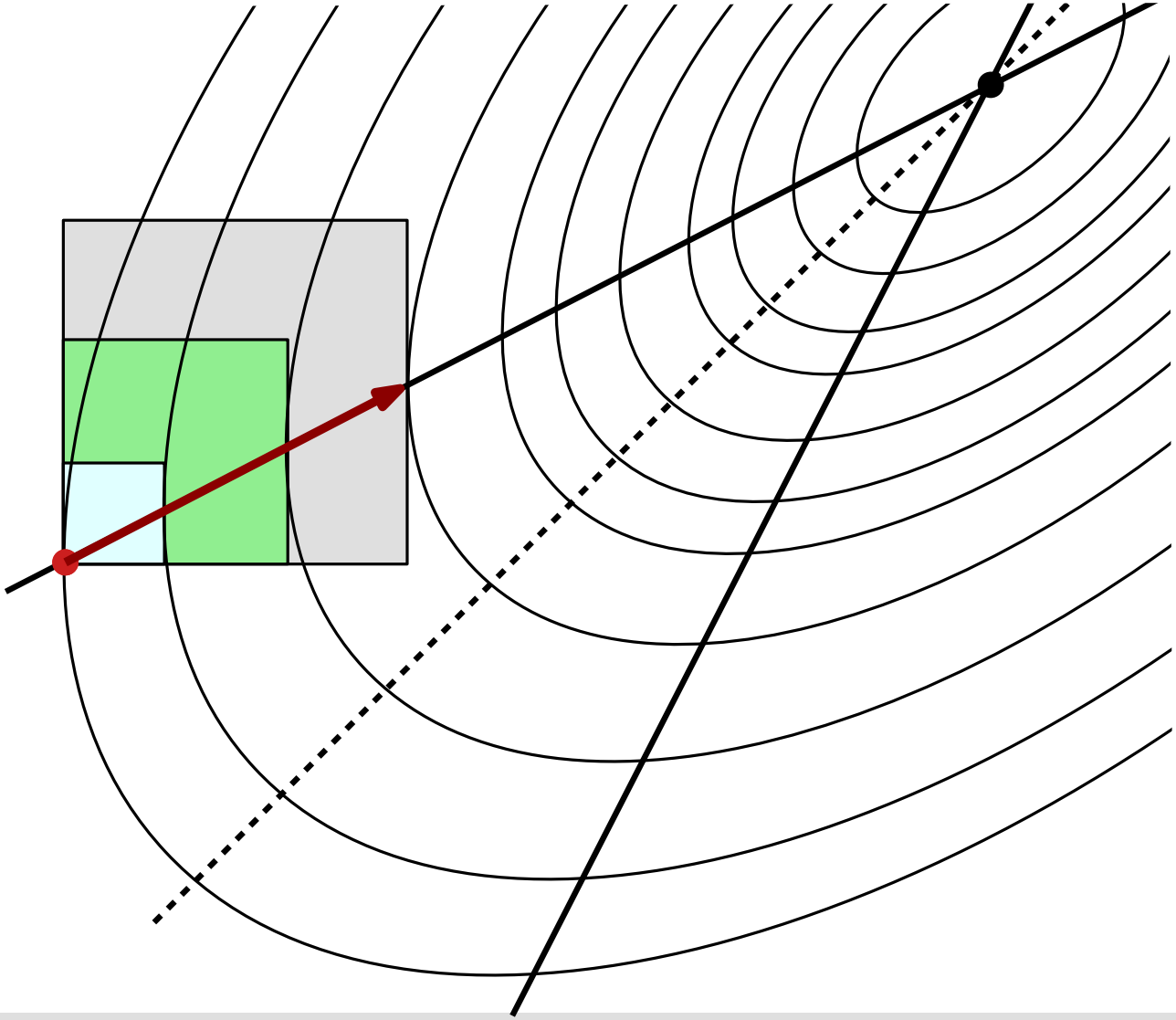
Lemma: A steepest descent path is a polygonal path with at most one bend before leaving a cell or reaching the minimum.



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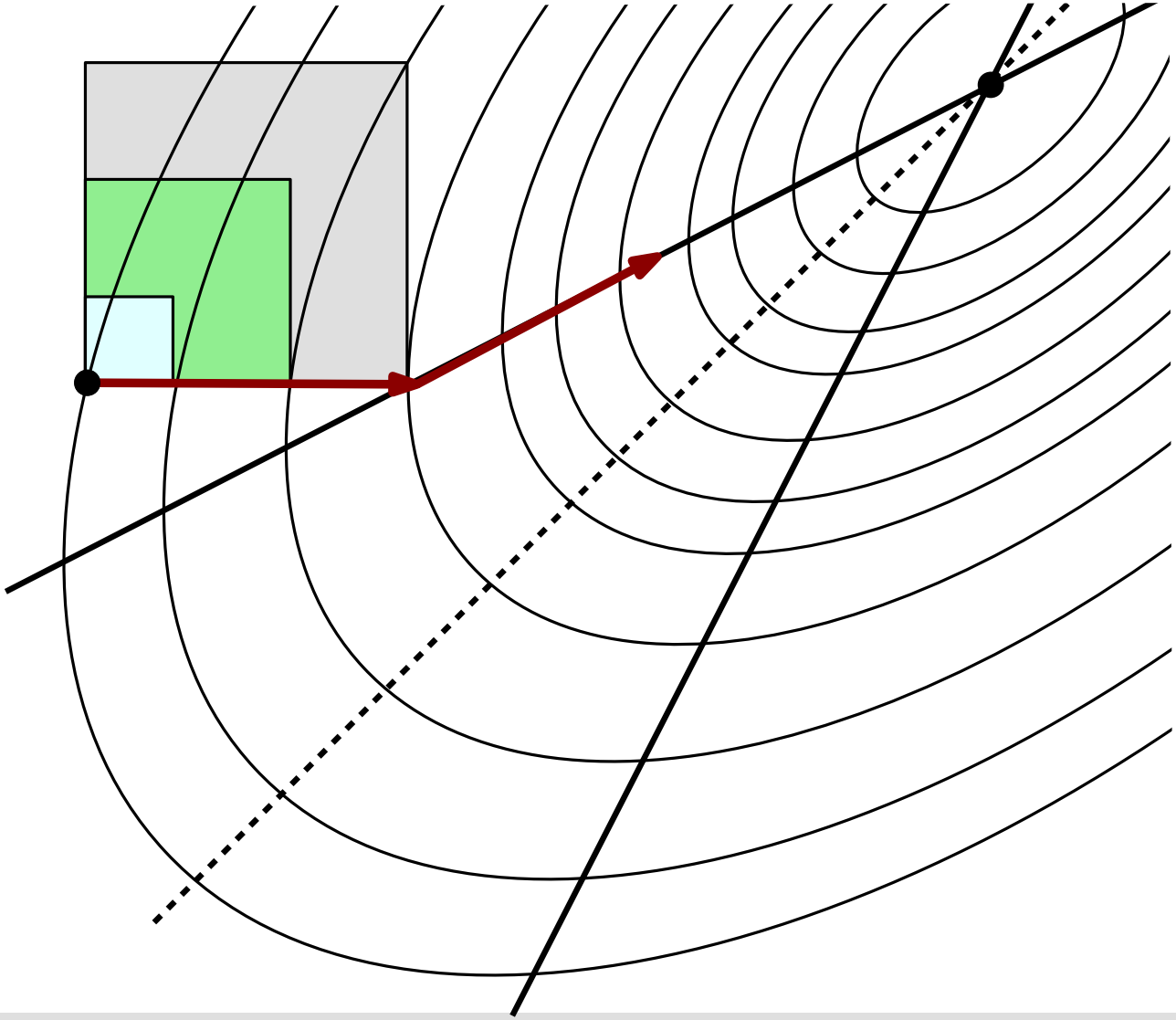
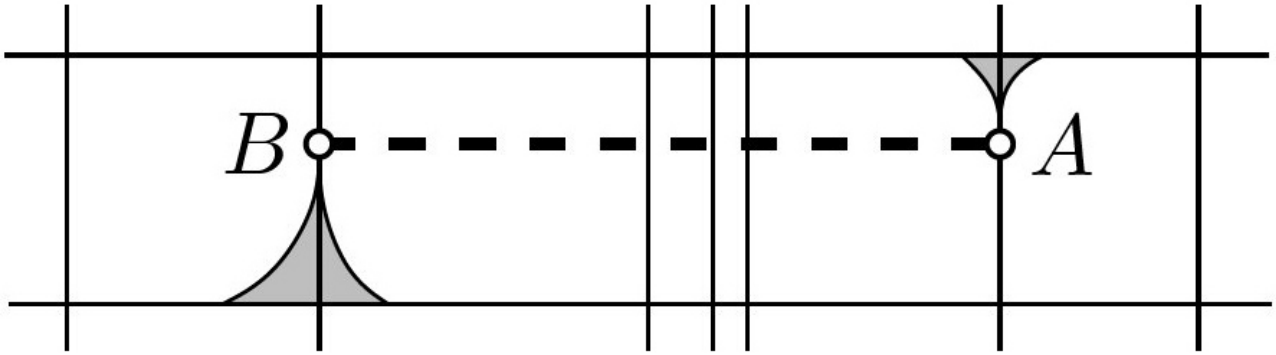
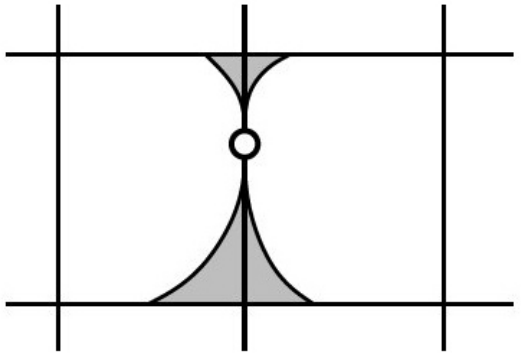


Critical Events

- known events

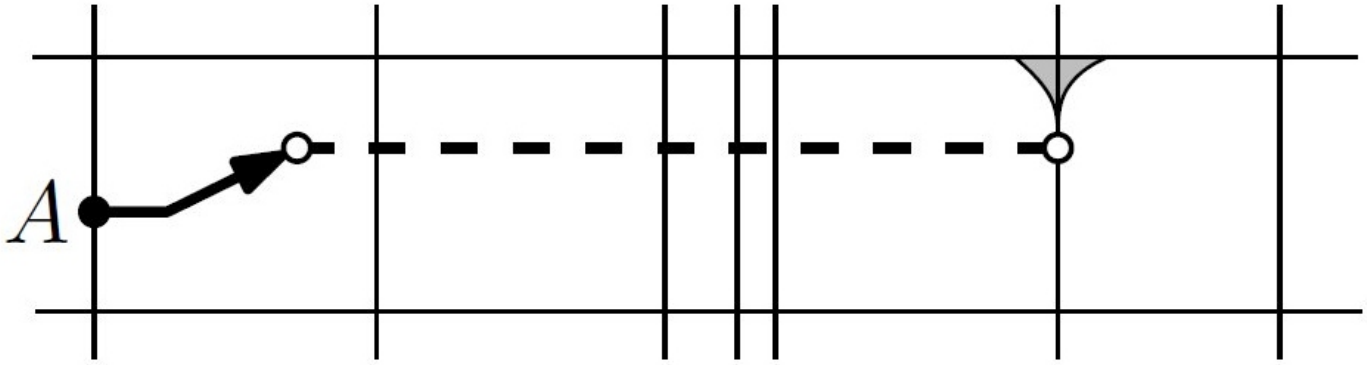


tunnel



boundary

- new events



$\rightarrow$ at any event the complexity of the curve is reduced!

Algorithm

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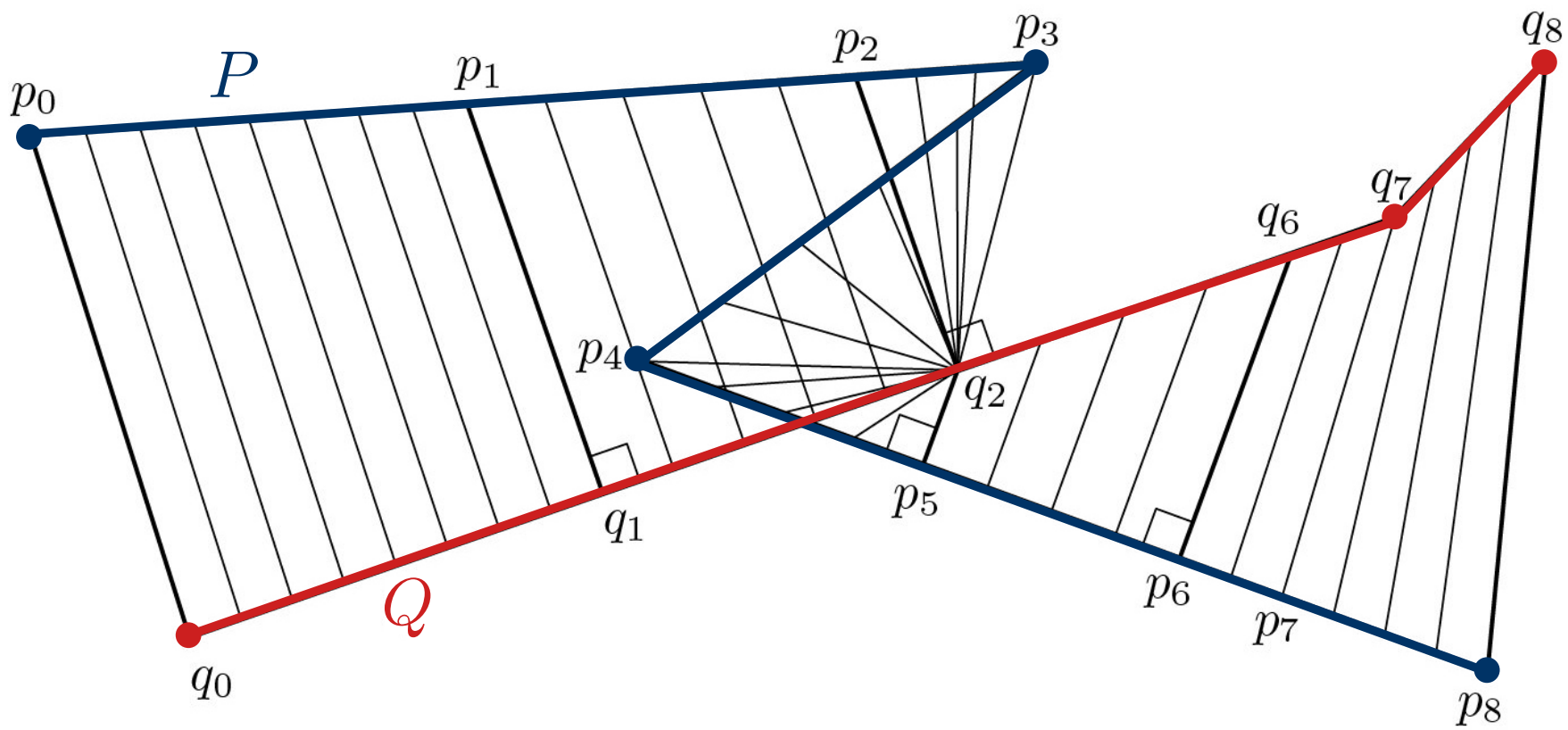
to find the optimal curve from A to B:

- follow steepest descent if possible
- watch for critical events

Theorem: A lexicographic Fréchet Matching can be computed in $O(mn(m+n)\log mn)$ time. [Rote, 2014]



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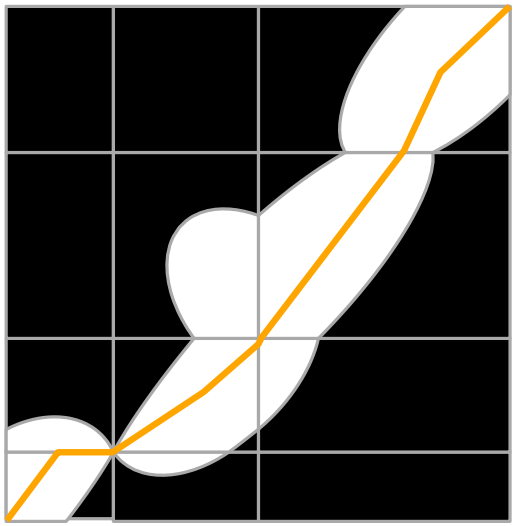

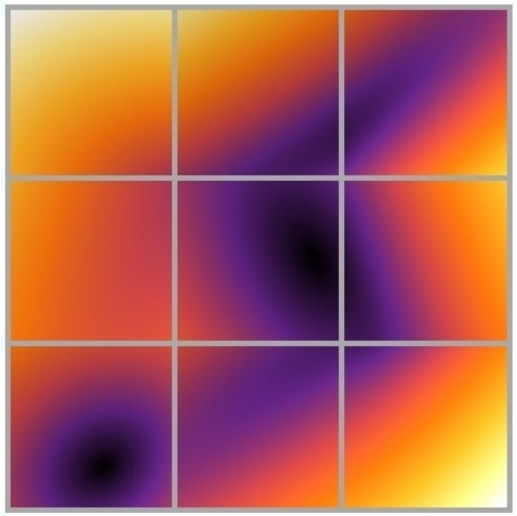

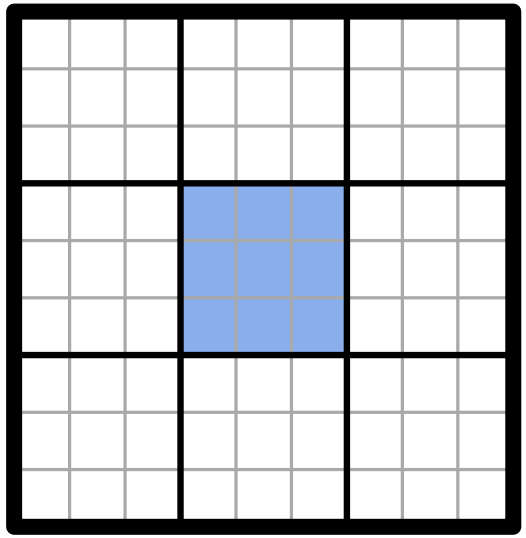

Searching Faster by Preprocessing

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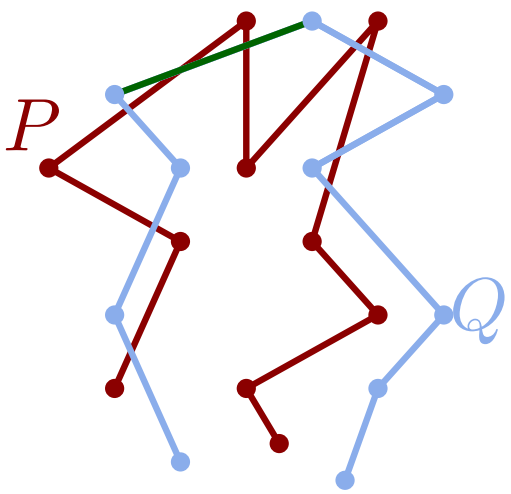
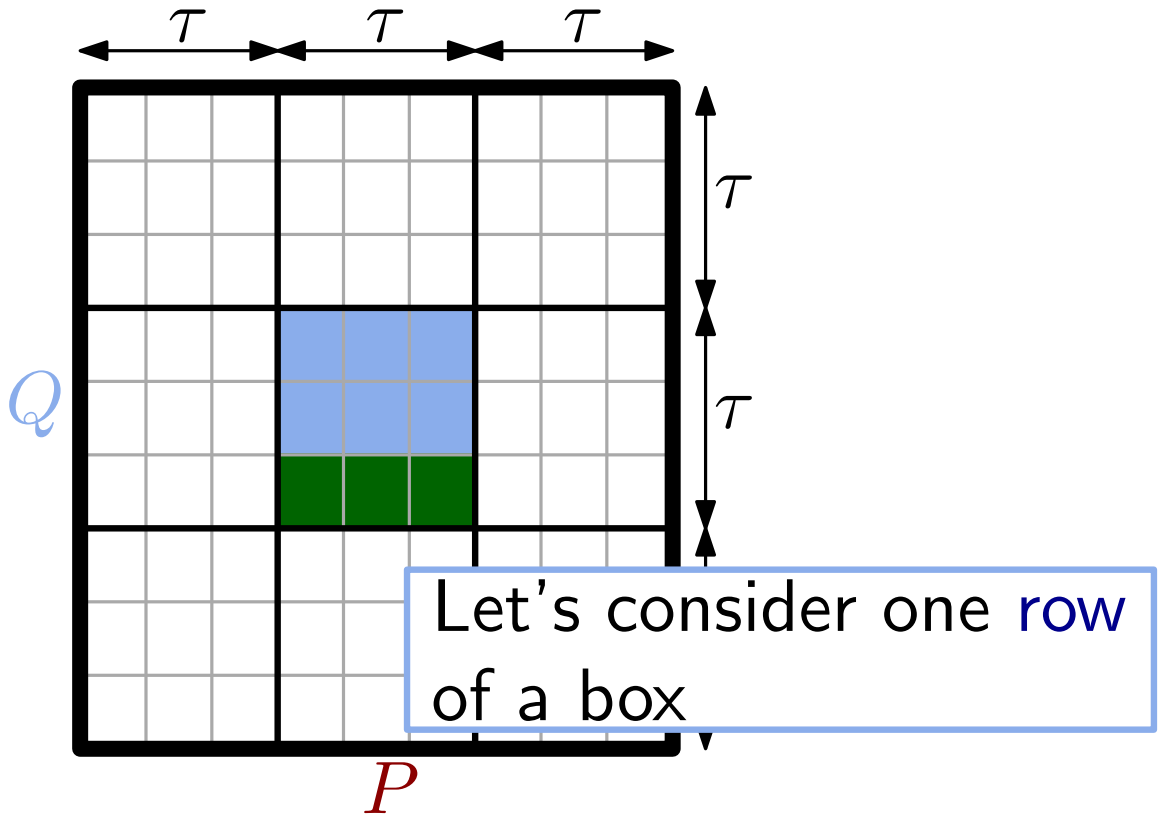
Can we preprocess to speed up the computation?

Elementary Boxes





going below $O(n^2)$

- process elementary boxes ($\tau \times \tau$ cells) in $o(\tau^2)$ time.
- How many combinatorial types of boxes are there?
- How can we find the combinatorial type of a box fast?

Door orders

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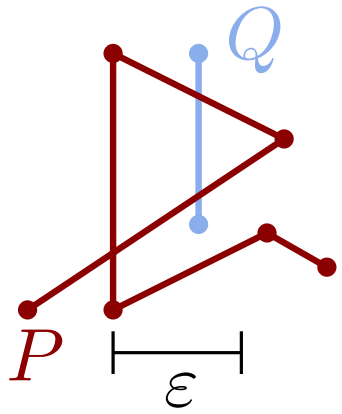
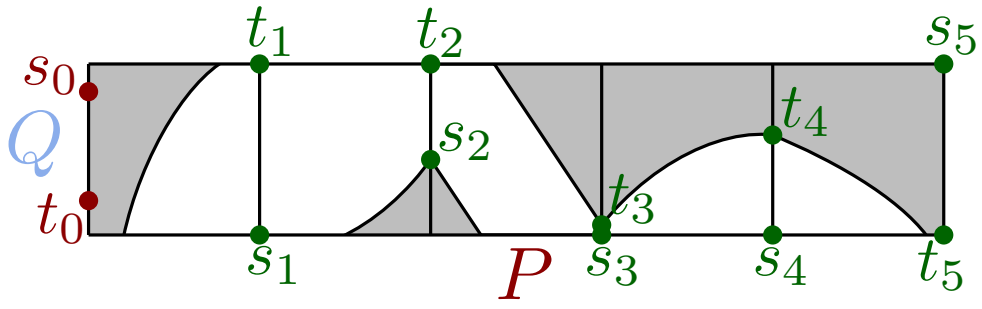




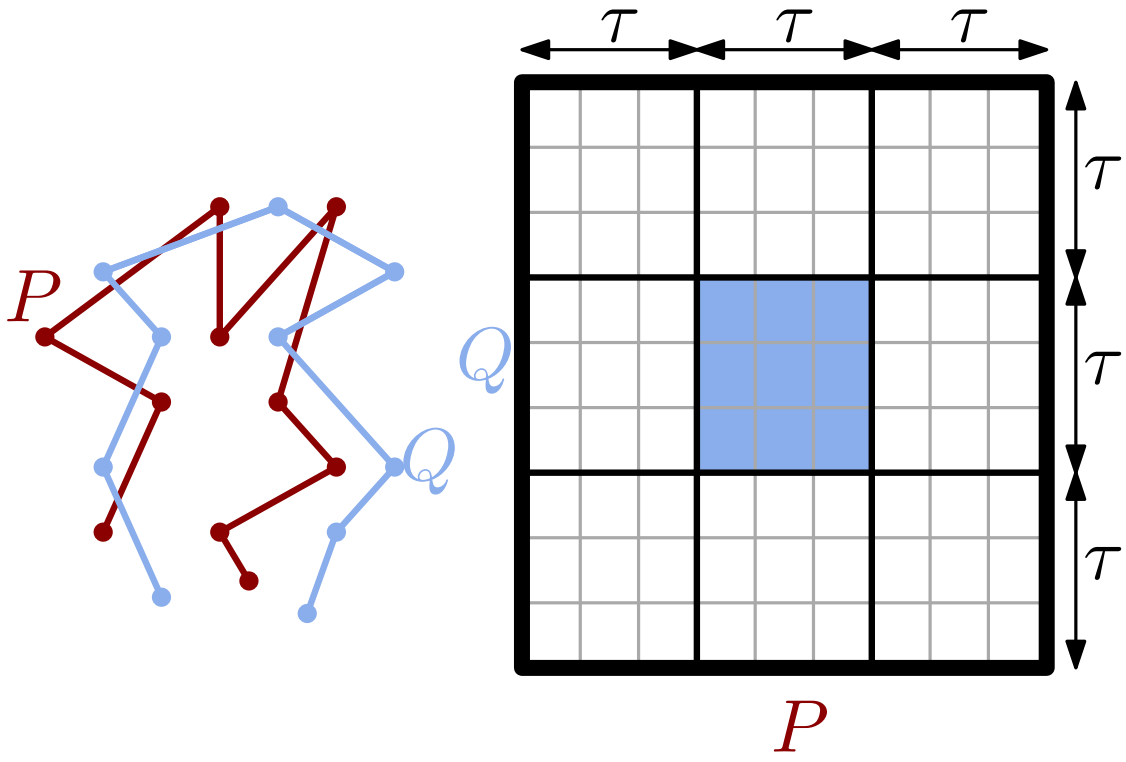
combinatorial type of a row

- white passages (*doors*): lower end (y-value) s_i, upper t_i
- s_0, t_0 encode reachability from the left (*reach-door*)
- row order: permutation of s_i, t_i (*sorted by height*)

$$s_1 s_3 s_4 t_5 t_3 t_0 s_2 t_4 s_0 s_5 t_1 t_2$$

- partial row order: permutation of s_i, t_i without s_0 and t_0
- row order determines how reachability propagates from left to right

Box signatures



How do we determine the signature?

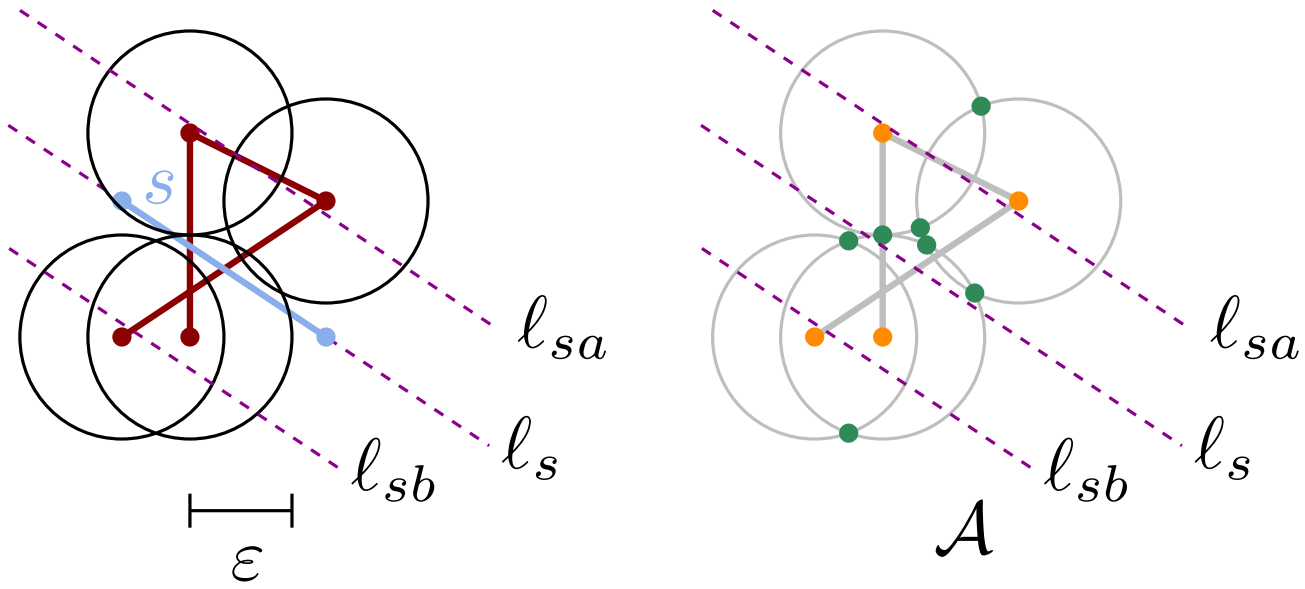
boils down to

- determine partial door orders
- add reach-doors

combinatorial type of an elementary box

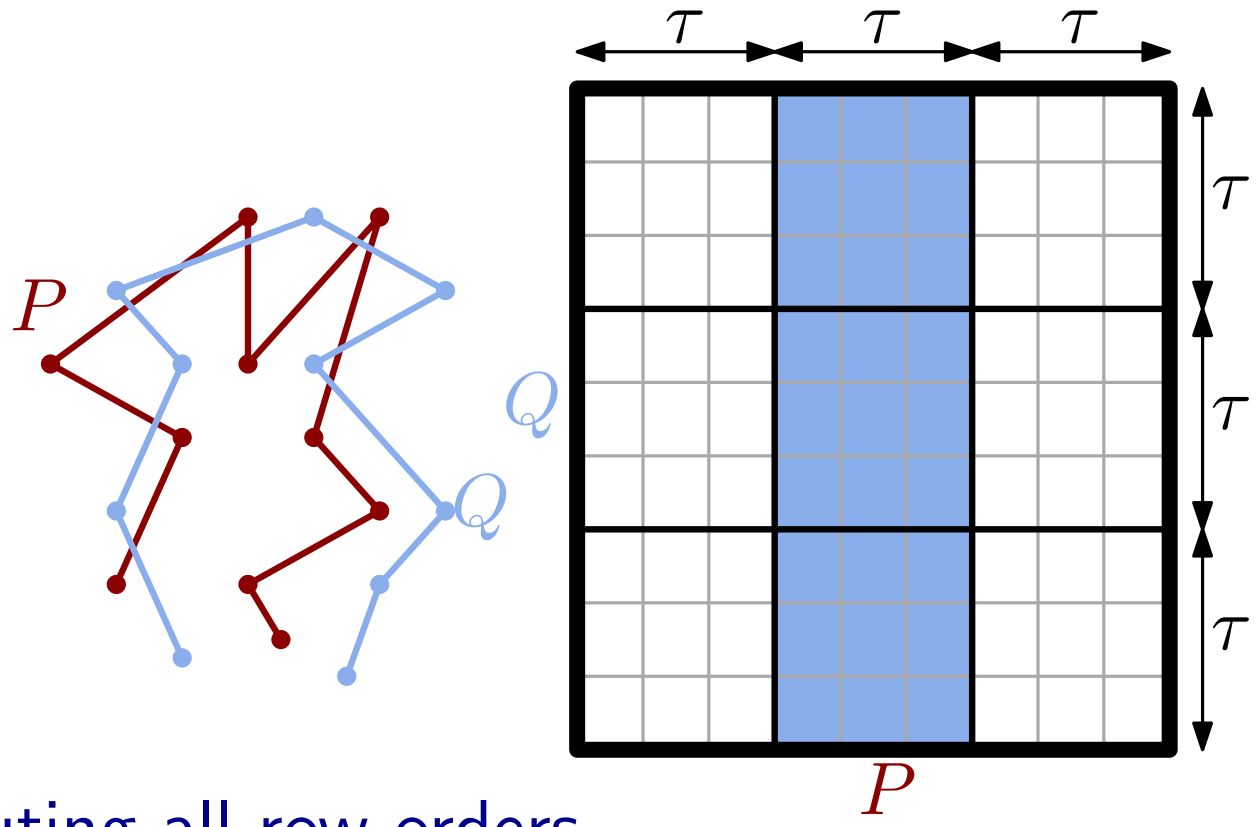
- signature of a box: all its row&column door orders
- partial signature of a box: all its partial door orders
- signature determines how reachability propagates from left&bottom to right&top

Querying the partial door order



- ordered sequence of intersections of s with ε-circles determines partial door order
- sequence of intersections of ℓ_s with circles is determined by position of ℓ_s, ℓ_{sa} and ℓ_{sb} relative to the circle centers and the intersection of circles
- with point location in the dual plane we can (after $O(\tau^c)$ preprocessing) query partial door orders in $O(\log \tau)$ time.

Putting things together

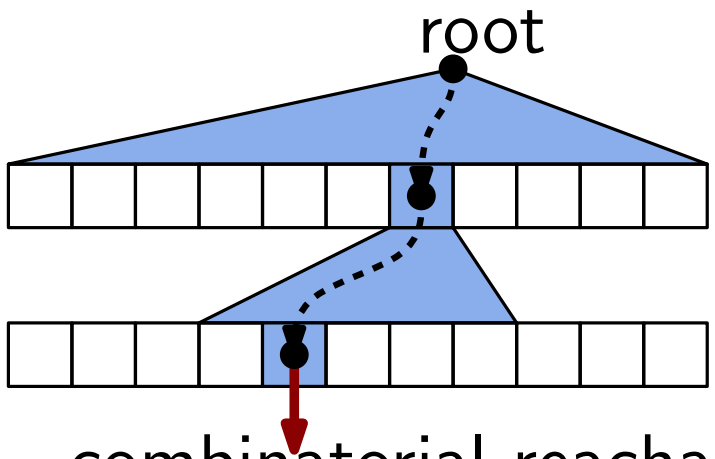


computing all row orders

- process vertical strips of elementary boxes
- compute partial row orders in strip in time $O(\tau^c + n \log \tau)$
- compute row orders from partial row orders in strip in time $O(n \log \tau)$ (having stored a suitable search structure)

compute all column orders in the same way

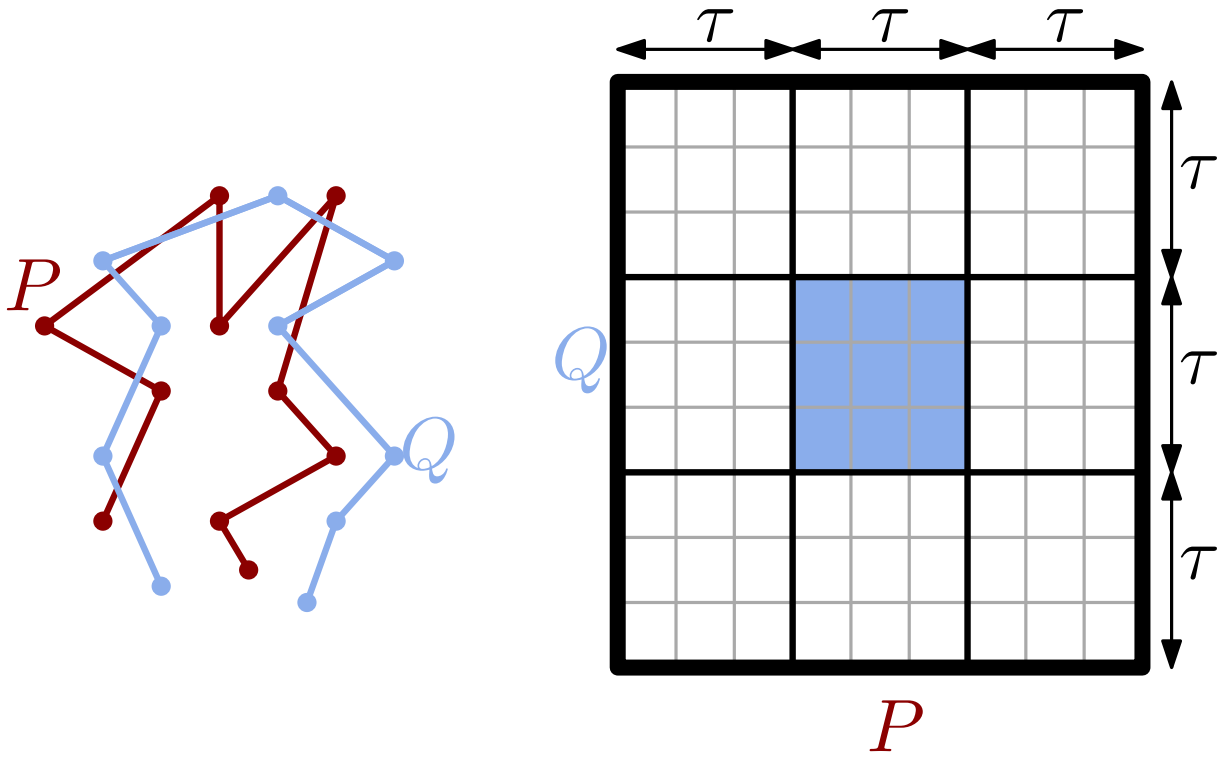
Decision Algorithm



compute partial door-orders
partial signature
add reach-doors
full signature

- Takes $O(n\tau^{c-1} + \frac{n^2 \log \tau}{\tau})$ time except for last step
- precompute lookup table to go from signature to reachability
- bottleneck:
  size of lookup table $=$ number of different signatures
- number of different row/column orders: $(2(\tau+1))!$
- number of signatures: $(2(\tau+1))!^{2\tau} = O(\tau^{O(\tau^2)})$
- Set $\tau = \Theta(\sqrt{\log n / \log\log n})$. Then ...

Decision Algorithm



Theorem: The Fréchet distance of two polygonal curves of size n each can be decided in $O(n^2(\log\log n)^{3/2}/\sqrt{\log n})$ time.

[Buchin, Buchin, Meulemans, Mulzer, 2014]

Summary

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many different ways to search the free space

- nicer following steepest descent
- faster using 4 Russians

Thanks!